Lecture 13. Virtual localization

- T-fixed point
- Computing $e_T(N^{vir})$.
- Aspinwall – Morrison formula

Reference :

Graber-Pandharipande Localization of virtual Classes

§1. T-fixed point.

- Torus action.

Let $T=(\mathbb{C}^*)^2$: 2 dim'l torus. $H^*_{BT}(pt)\cong\mathbb{Q}[\lambda_0.\lambda_1]$

Consider $T\curvearrowright V=\mathbb{C}^2$ defined by

$$(t_0,t_1)\cdot(z_0,z_1)=(t_0z_0,t_1z_1).$$

Let $X=\mathbb{P}^1=\mathbb{P}(V)$. Then $T\curvearrowright V$ lifts to a T-action on the univ. line bundle.

$$\mathcal{O}_{\mathbb{P}^1}(-1)=\{(\ell,v)\in\mathbb{P}^1\times V \mid v\in\ell\}\subset\mathbb{P}^1\times V.$$

Example 1: $H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1))=\mathbb{C}\langle\lambda_0\rangle\oplus\mathbb{C}\langle\lambda_1\rangle.$

$$H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(d))=\bigoplus_{\substack{a+b=d\\ a.b\geq 0}}\mathbb{C}\langle a\lambda_0+b\lambda_1\rangle$$

$T\curvearrowright\mathbb{P}^1$ has two fixed points $\overset{[0]}{\overset{\shortparallel}{p_0}}$ & $\overset{[\infty]}{\overset{\shortparallel}{p_1}}$

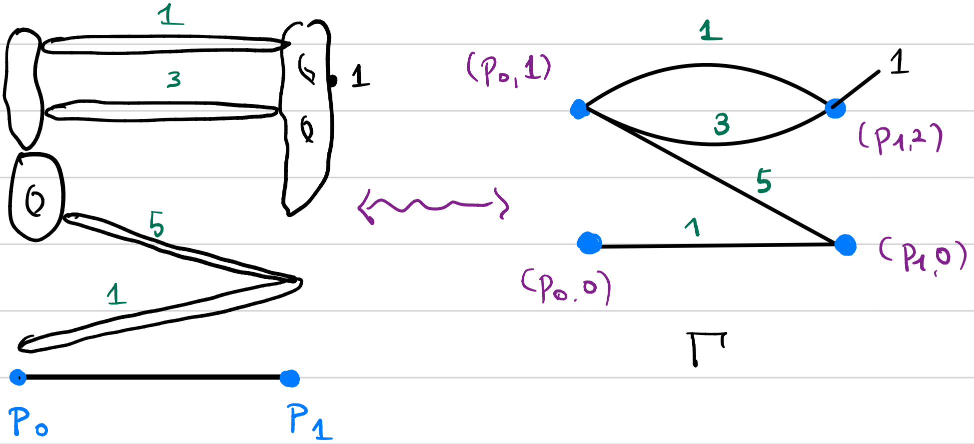
Example 2 $e_T(T_{p_0}\mathbb{P}^1)=\lambda_0-\lambda_1$, $e_T(T_{p_1}\mathbb{P}^1)=\lambda_1-\lambda_0$

(Hint: let $[\ell]\in\mathbb{P}^1$. Then $T_\ell\mathbb{P}^1\simeq\mathrm{Hom}(\ell,V/\ell)$).

○ Combinatorial data for T-fixed loci.

$T \curvearrowright \mathbb{P}^1 \rightsquigarrow T \curvearrowright \overline{M}_{g,n}(\mathbb{P}^1, d)$.

We remember combinatorial data of a connected component of T-fixed locus as follows:



- **Edge** : noncontracted components, labeled by the degree
- **Vertex** : connected component of $f^{-1}(p_0)$ or $f^{-1}(p_1)$.
  labeled by $(p_{i(v)}, g(v))$ $\quad i : V \longrightarrow \{0, 1\}$
  $g : V \longrightarrow \mathbb{Z}_{\geq 0}$
- **Leg** : markings
- **Flag** : $F = (e, v)$: incident edge-vertex pair

⚠ Γ is NOT a stable graph of the domain curve.

For each Γ, we associate

$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \quad \text{where } \overline{\mathcal{M}}_{0,1} = \overline{\mathcal{M}}_{0,2} := pt.$$

Then $\exists$ finite group A_Γ acting on $\overline{\mathcal{M}}_\Gamma$, where

$$1 \longrightarrow \underbrace{\prod_{e \in E(\Gamma)} \mathbb{Z}/d_e}_{\text{acts trivially on } \overline{\mathcal{M}}_\Gamma} \longrightarrow A_\Gamma \longrightarrow \mathrm{Aut}(\Gamma) \longrightarrow 1.$$

For each connected component of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$, we have

$$\begin{tikzcd} \gamma : \overline{\mathcal{M}}_\Gamma \arrow[rr] \arrow[dr] & & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \\ & \underbrace{[\overline{\mathcal{M}}_\Gamma / A_\Gamma]}_{\text{smooth DM stack!}} \arrow[ur, "\text{closed imbedding}"'] & \end{tikzcd}$$

Upshot Even if $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ is highly singular, T-fixed loci are smooth DM-stacks. This is an ideal situation to use virtual localization formula!

§2. Computing $e_T(N_\Gamma^{vir})$.

- $X = \mathbb{P}^1$.

Let $[f : (C, p_1, \dots, p_n) \longrightarrow X] \in \overline{\mathcal{M}}_{g,n}(X,d)^T$. Then we have

$$0 \to \underset{\substack{\uparrow \\ \text{Tangent space}}}{Def(f)} \to E_{0,f} \to E_{1,f} \to \underset{\substack{\uparrow \\ \text{obstruction space}}}{Obs(f)} \to 0$$

$$e_T(N^{vir}) := \frac{e(E_0^{mov})}{e(E_1^{mov})} = \frac{e(Def^{mov})}{e(Obs^{mov})}.$$

$Def(f)$ & $Obs(f)$ fit into the following long exact seq

$$\begin{tikzcd}
0 \arrow[r] & Ext^0(\Omega_C(D), \mathcal{O}_C) \arrow[r] & H^0(C, f^*TX) \arrow[r] & Def(f) \arrow[dll, out=-30, in=150] \\
 & Ext^1(\Omega_C(D), \mathcal{O}_C) \arrow[r] & H^1(C, f^*TX) \arrow[r] & Obs(f) \arrow[r] & 0
\end{tikzcd} \quad (*)$$

- $D = p_1 + \dots + p_n$
- $Ext^0(\Omega_C(D), \mathcal{O}_C) \cong T_{[e]} Aut(C, D)$.

Few Remarks :

* Canonical T action on $Def(f)$ & $Obs(f)$ naturally extends to $(*)$. $(*)$ = exact sequence of T-representations

* In this case, $[\overline{\mathcal{M}}_\Gamma / A_\Gamma]^{vir} = [\overline{\mathcal{M}}_\Gamma / A_\Gamma]$.

* It is easier to compute the moving part of (*) after pulling back to $\overline{\mathcal{M}}_\Gamma$. The price to pay: we have to divide the order of A_Γ at the end.

$$e(N_\Gamma^{vir}) = \frac{e(H^0(C, f^*TX)^{mov})\; e(Ext^1(\Omega_C(D), \mathcal{O}_C)^{mov})}{e(H^1(C, f^*TX)^{mov})\; e(Ext^0(\Omega_C(D), \mathcal{O}_C)^{mov})}$$

⚠

We only wrote a fiber of sheaves over a point in $\overline{\mathcal{M}}_\Gamma$. The point here is that those spaces glue naturally & gives T-equiv. coherent sheaves on $\overline{\mathcal{M}}_\Gamma$.

(I) Moving parts from deforming f.

Let's compute $H^\bullet(C, f^*TX)^{mov}$. Consider a partial normalization of C forced by Γ.

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \bigoplus_{v\in V} \mathcal{O}_{C_v} \oplus \bigoplus_{e\in E} \mathcal{O}_{C_e} \longrightarrow \bigoplus_{F} \mathbb{C}_{x_F} \longrightarrow 0$$

Tensor with f^*TX. and take $H^\bullet$

$$\begin{tikzcd} 0 \arrow[r] & \underline{H^0(C, f^*TX)} \arrow[r] & \bigoplus_{v} H^0(C_v, f^*TX) \oplus \bigoplus_{e} H^0(C_e, f^*TX) \arrow[r] & \bigoplus_{F} T_{x_F}X \arrow[dll] \\ & \underline{H^1(C, f^*TX)} \arrow[r] & \bigoplus_{v} H^1(C_v, f^*TX) \arrow[r] & 0 \end{tikzcd}$$

- Contribution from $H^0(C_e, f^*TX)$
  ↳ trivial bundle with nontrivial weights

Recall: $f|_{C_e} : \underset{\mathbb{P}^1}{\underset{\cong}{C_e}} \longrightarrow \mathbb{P}^1, \quad z \longmapsto z^{d_e}$

Euler sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2 \longrightarrow T\mathbb{P}^1 \longrightarrow 0$$

Pulling back to C_e : & $H^{\bullet}$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] & H^0(\mathcal{O}(d_e)) \arrow[r, phantom, "\otimes"] & \mathbb{C}^2 \arrow[r] & H^0(f^*T\mathbb{P}^1) \arrow[r] & 0 \\
& \mathrm{wt}=0 \arrow[u] & \mathrm{wt}=\frac{a}{d_e}\lambda_0+\frac{b}{d_e}\lambda_1 \arrow[u] & \mathrm{wt}=-\lambda_0,-\lambda_1 \arrow[u] & &
\end{tikzcd}$$

$a+b=d_e$

$$\Rightarrow e_T\left(H^0(f^*T\mathbb{P}^1)^{mov}\right)$$

$$=\prod_{a=0}^{d_e-1}(d_e-a)\cdot\left(\frac{\lambda_0-\lambda_1}{d_e}\right)\bullet\prod_{a=0}^{d_e-1}(d_e-a)\left(\frac{\lambda_1-\lambda_0}{d_e}\right)$$

$$=(-1)^{d_e}\frac{(d_e!)^2}{d_e^{2d_e}}(\lambda_0-\lambda_1)^{2d_e}$$

- Contribution from $H^1(C_v, f^*TX)$

C_v is a contracted component. So

$$H^1(C_v, f^*TX) = H^1(\mathcal{O}_{C_v}) \otimes T_{i(v)}X.$$

$$= \mathbb{E}^\vee_{[\mathcal{O}_C]} \otimes T_{i(v)}X.$$

$\uparrow$ wt $= \lambda_{i(v)} - \lambda_{i(v')}$

Exercise For a vector bundle E of rk$=r$, let

$$C_t(E) := 1 + t c_1(E) + \cdots + t^r c_r(E)$$

be the Chern polynomial of E (t is a formal variable)

Let L be a line bundle. Then show

$$C_t(E \otimes L) = \sum_{r=0}^{r} t^i c_1(L)^{r-i} c_i(E).$$

$$\Rightarrow e_T(H^1(C_v, f^*T\mathbb{P}^1))$$

$$= C_{(\lambda_{i(v)} - \lambda_{i(v')})^{-1}}(\mathbb{E}^\vee_v) \cdot (\lambda_{i(v)} - \lambda_{i(v')})^{g(v)}$$

- Contribution from $H^0(C_v, TX)$, $T_{x_F}X$.

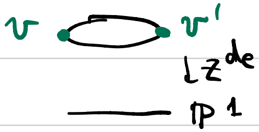
$H^0(C_v, TX) = T_{i(v)}X \leftarrow$ wt$= \lambda_{i(v)} - \lambda_{i(v')}$

$T_{x_F}X \quad \leftarrow$ wt $= \lambda_v - \lambda_{v'} \qquad F = (e, v).$

(II) Moving parts from deforming C.

Let $F=(e,v)$ be a flag associated to

$$C_e \xrightarrow{d_e} \mathbb{P}^1$$



Denote

$$\omega_F = e_T(T_v C_e) = \frac{\lambda_{i(v)} - \lambda_{i(v')}}{d_e}$$

We use partial normalization of C forced by C. The moving part only appears at flags!

- Contribution from $\mathrm{Ext}^0(\Omega_C(D), \mathcal{O}_C)$

(i) If C has two special points ⇒ $\mathrm{Ext}^{0,\mathrm{mov}} = 0$

(ii) If C has one special point ⇒ $e_T(\mathrm{Ext}^{0,\mathrm{mov}}) = \omega_F$

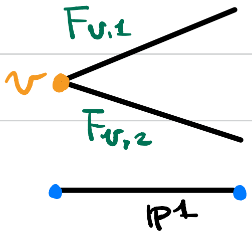
- Contribution from $\mathrm{Ext}^1(\Omega_C(D), \mathcal{O}_C)$

(i) If $F=(e,v)$ connects a contracted component & a noncontracted component.

$\Rightarrow e_T(\cdots) = \omega_F - \psi_F$

(ii) If $F=(e,v)$ connectes two noncontracted components

$\Rightarrow e_T(\cdots) = \omega_{F_{v,1}} + \omega_{F_{v,2}}$



Summary

$$\frac{1}{e_T(N_\Gamma^{vir})} = \prod_{\substack{F=(e,v)\\ v=\text{stable}}} \frac{1}{\omega_F - \psi_F} \prod_F (\lambda_{i(v)} - \lambda_{i(v')})$$

$$\prod_{\substack{v\in V(\Gamma)\\ v=\text{stable}}} c_{(\lambda_{i(v)}-\lambda_{i(v')})^{-1}}(E_v^\vee) \cdot (\lambda_{i(v)} - \lambda_{i(v')})^{g(v)-1}$$

$$\prod_{\substack{n(v)=2\\ g(v)=0}} \frac{1}{\omega_{F_{v,1}} + \omega_{F_{v,2}}} \prod_{\substack{n(v)=1\\ g(v)=0}} \omega_F$$

$$\prod_{e\in E(\Gamma)} \frac{(-1)^{d_e} d_e^{2d_e}}{(d_e!)^2 (\lambda_0 - \lambda_1)^{2d_e}}$$

Cor Let $\pi: \overline{M}_{g,n}(\mathbb{P}^1, d) \longrightarrow \overline{M}_{g,n}$. $\alpha_i \in H^*(\mathbb{P}^1)$. Then

$$\pi_*\left(\prod_{i=1}^{n} ev_i^*\alpha \cap [\overline{M}_{g,n}(\mathbb{P}^1, d)]^{vir}\right) \in R^*(\overline{M}_{g,n}).$$

§3. Aspinwall – Morrison formula

Let $Y = \mathrm{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\right)$. ← quasi-projective Calabi-Yau 3 fold.

$\Rightarrow$ $\mathrm{vdim}\, \overline{\mathcal{M}}_g(Y, d) = 0$

By the negativity of $\mathcal{O}_{\mathbb{P}^1}(-1)$,

$$\overline{\mathcal{M}}_g(Y, d) \cong \overline{\mathcal{M}}_g(\mathbb{P}^1, d)$$

and.

$$[\overline{\mathcal{M}}_g(Y, d)]^{vir} = e(\mathrm{Obs}) \cap [\overline{\mathcal{M}}_g(\mathbb{P}^1, d)]^{vir}.$$

$$\mathrm{Obs} = R^1\pi_* f^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$$

$$\begin{tikzcd}
 & \mathcal{O}(-1) \oplus \mathcal{O}(-1) \arrow[d] \\
\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1, d) \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{P}^1 \\
\overline{\mathcal{M}}_g(\mathbb{P}^1, d) &
\end{tikzcd}$$

Exercise Let $d \geq 1$. Then $R^1\pi_* f^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$ is a locally free sheaf of rank $= 2g-2+2d$.

Let

$$C(g, d) = \int_{[\overline{\mathcal{M}}_g(\mathbb{P}^1, d)]^{vir}} e(\mathrm{Obs}) \in \mathbb{Q}$$

From string theory in HEP, $C(g.d)$ should satisfies certain property (multiple cover formula) ie

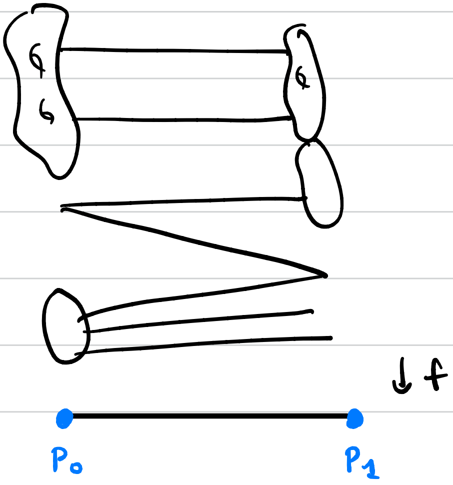
$$C(g.d) = d^{2g-3} C(g.1)$$

Thm (Faber-Pandharipande) Let $d \geq 1$. Then

$$C(g.d) = \begin{cases} d^{-3} & \text{if } g=0 \\ \dfrac{|B_{2g}|\, d^{2g-3}}{2g\,(2g-3)!} & \text{if } g>0 \end{cases}.$$

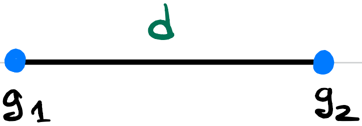
Let's compute $C(g.d)$ by the virtual localization.

First obstacle T-fixed loci are very complicated as $g \,\&\, d \to \infty$.



Idea Two different equivariant lifts of $\mathcal{O}_{\mathbb{P}^1}(-1)$ cancel all contributions of $\overline{\mathcal{M}}_\Gamma$ except for those Γ of the form

$\Gamma =$



Let $H^*(\mathbb{P}^1) \simeq \mathbb{Q}[H]/H^2$. Choose two equiv lifs:

$$e_T(\mathcal{O}(-1) \otimes \lambda_1) = -H + \lambda_1$$

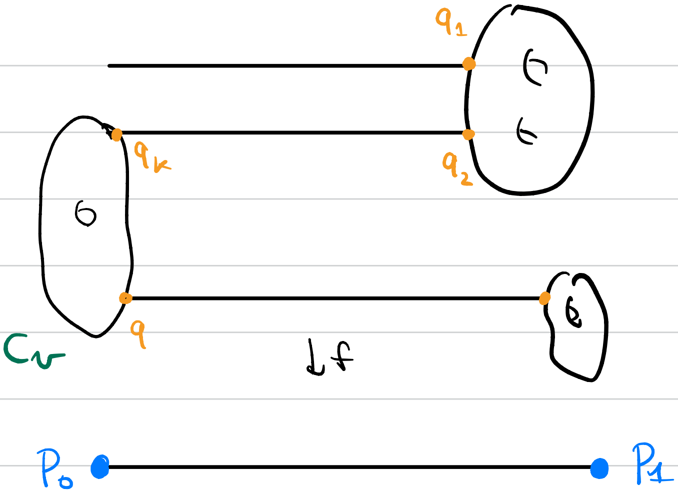
$$e_T(\mathcal{O}(-1) \otimes \lambda_0) = -H + \lambda_0 .$$

Lemma Let Γ correspond to a connected component of $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)^T$. If Γ contains a vertex v of $n(v) \geq 2$, then the contribution vanishes.

pf) Let $[f : C \longrightarrow \mathbb{P}^1] \in [\overline{\mathcal{M}}_\Gamma / A_\Gamma] \hookrightarrow \overline{\mathcal{M}}_g(\mathbb{P}^1, d)^T$.

Consider the partial normalization of C by Γ.

$$\begin{tikzcd}
\tilde{C} \arrow[r, "\alpha"] \arrow[d, equal] & C \\
\bigsqcup \tilde{C}_i &
\end{tikzcd}$$



Suppose $\exists\ v \in V(\Gamma)$ s.t $n(v)=2$. Let $i(v) = p_0$.

Consider the exact sequence

$$0 \to \mathcal{O}_C \longrightarrow \alpha_* \mathcal{O}_{\tilde{C}} \longrightarrow \bigoplus_q \mathbb{C}_q \to 0$$

Tensor with $f^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \lambda_0$ and take H^* :

$$0 \to \bigoplus H^0(\tilde{C}_i, f^*\mathcal{O}(-1)\otimes\lambda_0) \longrightarrow \bigoplus_q H^0(q, f^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\lambda_0) \to H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\lambda_0)$$

Since $\tilde{C}_v$ contracts to a point p_0,

- $$H^0(\tilde{C}_v, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \cong H^0(\mathcal{O}_{\tilde{C}_v}) \otimes \underbrace{\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\lambda_0\big|_{p_0}}_{wt = -\lambda_0+\lambda_0=0}$$

- $$\bigoplus_{q \vdash v} H^0(q, f^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\lambda_0) \leftarrow$$ dim $= n(v)$ of weight 0 spaces

If $n(v) \geq 2 \Rightarrow R^1\pi_* f^*(\mathcal{O}(-1)\oplus\mathcal{O}(-1))$ contains trivial T-repns.

$$\Rightarrow e_T(\mathrm{Obs}) = 0$$

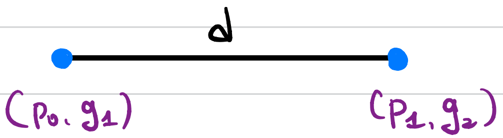
∎

⚠ If you take different T-equiv lift of $\mathcal{O}(-1)$s, then you will get much complicated T-fixed loci (see [GP]).

Result Possible configurations of Γ:

$(g_1+g_2=g)$.

$\Gamma_{g_1g_2} =$



By the partial normalization sequence, we have

$$e_T(Obs) = (-1)^{d-1}\frac{(d!)^2}{d^{2d}}(\lambda_0-\lambda_1)^{2d-2}$$

$$\cdot (-1)^{g_1}\lambda_{g_1}\cdot c_{(\lambda_1-\lambda_0)^{-1}}(\mathbb{E}^\vee)(\lambda_1-\lambda_0)^{g_1}$$

$$\cdot (-1)^{g_2}\lambda_{g_2}\cdot c_{(\lambda_0-\lambda_1)^{-1}}(\mathbb{E}^\vee)(\lambda_0-\lambda_1)^{g_2}$$

$\uparrow$

$\deg = 2g-2+2d$

By the virtual localization, we conclude

$$C(g,d) = \sum_{g_1+g_2=g}\boxed{\frac{1}{d}}\,\mathrm{Cont}(\Gamma_{g_1,g_2})$$

$\uparrow$ Automorphism factor

$$\mathrm{Cont}(\Gamma_{g_1 g_2}) = (-1)^{d-1}\frac{(d!)^2}{d^{2d}}(\lambda_0-\lambda_1)^{2d-2+g}\,\lambda_{g_1}(-1)^{g_2}\lambda_{g_2} \leftarrow e_T(Obs)$$

$$\cdot\frac{d^{2g_1-1}}{(\lambda_0-\lambda_1)^{2g_1-1}}\,\psi_1^{2g_1-2}\cdot(\lambda_0-\lambda_1)^{g_1} \leftarrow \text{vertex } g_1$$

$$\cdot\frac{d^{2g_2-1}}{(\lambda_1-\lambda_0)^{2g_2-1}}\,\psi_2^{2g_2-2}\cdot(\lambda_1-\lambda_0)^{g_2} \leftarrow \text{vertex } g_2$$

$$\cdot\frac{(-1)^d d^{2d}}{(d!)^2(\lambda_0-\lambda_1)^{2d}} \qquad \leftarrow \text{edge } e$$

Other terms vanish because of the dimension reason.
After integrating over $\overline{M}\Gamma_{g_1g_2} = \overline{M}_{g_1,1} \times \overline{M}_{g_2,1}$, we have

$$C(g,d) = \underbrace{\frac{1}{d} \cdot \frac{d^{2g-2+2d}}{d^{2d}}}_{d^{2g-3}} \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 0}} b_{g_1} b_{g_2}$$

where

$$b_g = \begin{cases} 1 & \text{if } g=0 \\ \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g & g>0 \end{cases}$$

We are reduced to compute b_g. The general formula is given in

Faber-Pandharipande, Hodge integrals & GW theory

Exercise Use Pixton's formula to compute b_g when $g=1,2$. Check whether the result coincides with our formula