Lecture 2

24.09.2020

Today.

- Basic theory of divisors on a curve
  - Weil vs Cartier
  - pullback of divisors

- AJ theorem
  - Picard group
  - Abel-Jacobi theorem for elliptic curves.

§1. Basic theory of divisors.

Divisor = " $\operatorname{codim}_{\mathbb{C}} = 1$ subspace of X "

(1.1) Weil divisor

X : connected, nonsingular, complete curve / $\mathbb{C}$

$\Rightarrow$ $x \in X$ closed point $\mathcal{O}_{X,x}$ is DVR.

Def (Weil divisor)

- $\operatorname{Div}(X) = \left\{ \sum n_i p_i \;\middle|\; p_i \in C \text{ closed pt}, n_i \in \mathbb{Z} \right\}$
- $\deg : \operatorname{Div}(X) \longrightarrow \mathbb{Z}$. $\sum n_i p_i \longmapsto \sum n_i$

↳ Huge

Let $K(X)$ = function field of X. Then there is a canonical isomorphism

$$K(X) \cong \operatorname{Frac}(\mathcal{O}_{X,x}).$$

Let $\nu_p : K(X) \to \mathbb{Z}$ be the valuation.

$$(f) := \sum_{p \in X} \nu_p(f) \cdot p$$

"leading order of f near p"

Def If $D = (f)$, it is called a <u>principal divisor</u>

Def $A_0(X) := \mathrm{Div}(X)/\sim$ $D \sim D'$ if $D - D' = (f)$.

↳ This is still huge

(1.2) Cartier divisor.

↳ local defining equation.

Let K_X^* = sheaf of rational functions on X.
(so it is the constant sheaf of $K(X)$)
We have an exact seq.

$$0 \to \mathcal{O}_X^* \longrightarrow K_X^* \longrightarrow K_X^* / \mathcal{O}_X^* \to 0$$

(*)

$$\begin{array}{ccccccccc} 0 \to & H^0(\mathcal{O}_X^*) & \to & H^0(K_X^*) & \longrightarrow & H^0(K_X^*/\mathcal{O}_X^*) & \xrightarrow{\partial} & H^1(\mathcal{O}_X^*) & \to 0 \\ & \| & & \| & & & & \| & \\ & \mathbb{C}^* & & K(X) & & & & \mathrm{Pic}(X) & \end{array}$$

Def A Cartier divisor is an element of

$$D \in H^0(X, \mathcal{K}_X^* / \mathcal{O}_X^*).$$

$D = \{(U_i, f_i)\}$, where $\{U_i\}$: open cover of X.

$f_i \in \mathcal{K}^*(U_i)$ s.t $f_i / f_j \in \mathcal{O}^*(U_i \cap U_j)$.

Def D is a principal divisor if it is an image of

$$H^0(\mathcal{K}_X^*) \to H^0(\mathcal{K}_X^* / \mathcal{O}_X^*).$$

From the sequence (*)

$$\mathrm{Pic}(X) \cong H^0(\mathcal{K}_X^* / \mathcal{O}_X^*) \big/ H^0(\mathcal{K}_X^*)$$

In practice, for $D = \{(U_i, f_i)\} \in H^0(\mathcal{K}_X^* / \mathcal{O}_X^*)$.

$$\mathcal{O}(D)|_{U_i} := f_i^{-1} \mathcal{O}_X|_{U_i} \subset \mathcal{K}^*(U_i).$$

(1.3) Comparison.

We have a map

$$H^0(\mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow \mathrm{Div}(X)$$

sending

$$D = \{(U_i, f_i)\} \longmapsto \sum v_p(f_i)\, p \,.$$

This is well-defined because if $p \in U_i \cap U_j$,

$$f_i/f_j \in \mathcal{O}^*(U_i \cap U_j) \Longrightarrow v_p(f_i) = v_p(f_j) \,.$$

Moreover it sends (principal divisor) $\to$ (principal divisor), hence

$$\mathrm{Pic}(X) \longrightarrow A_0(X)$$

<u>Prop</u> The comparison map is an isomorphism.

Pf) [Hartshorne, II.6.11] $\boxtimes$

How this works?

Let $p \in X$. We associate a line bundle $\mathcal{O}(p)$:
for $U \subseteq X$ open,

$$\mathcal{O}(p)(U) = \{ f \in K^*(U) \mid v_x(f) + \delta_p(x) \geq 0, \ \forall x \in U \} \cup \{0\}$$

here $\delta_p(x) = \begin{cases} 1 & \text{if } p = x \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow$ $\mathcal{O}(p)$ is an invertible sheaf. ($\mathcal{O}(p)^\vee \cong \mathcal{O}(-p)$).

If $D = \sum_{i=1}^m n_i p_i \in \mathrm{Div}(X)$, we take

$$\mathcal{O}(D) := \bigotimes_{i=1}^m \mathcal{O}(p_i)^{\otimes n_i}.$$

Let's say little more about $\mathcal{O}(p)$:
It has a canonical section $s_p \in H^0(\mathcal{O}_X(p))$
s.t $(s_p) = p$.

$T \cdot$: $s_p \in H^0(\mathcal{O}_X(p)) \Leftrightarrow s_p : \mathcal{O}_X \to \mathcal{O}_X(p)$

Any $f \in \mathcal{O}_X(U)$ satisfies $v_x(f) + \delta_p(x) \geq 0$ $\lrcorner$

ASIDE: Degree of a line bundle

We saw two ways to compute $\deg L$.

① Choose a nontrivial meromorphic section s of L. Then

$$\deg L = (s) \qquad \in \mathbb{Z}$$

② Use the exponential sequence:

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

$$\deg L := c_1(L) \cap [X] \in \underset{\mathbb{Z}}{\underset{\text{\rotatebox{90}{$\cong$}}}{H_0(X, \mathbb{Z})}}$$

CLAIM ① = ②

One way to prove this statement is to use

$$c_1(L) = e(L_{\mathbb{R}}) \in H^2(X, \mathbb{Z})$$

$L_{\mathbb{R}}$: real oriented $\text{rank}_{\mathbb{R}} = 2$ vector bundle

$e(L_{\mathbb{R}})$: Euler class of $L_{\mathbb{R}}$

The proof almost immediately follows if you know the definition of the Euler class (& Thom isom).

<u>HW</u> Justify <u>CLAIM</u> by yourself.
↳ See [Bott - Tu]

(1.4) Pullback of divisors.

$f: X \to Y$ nontrivial morphism btw curves $/\mathbb{C}$

<u>Def</u> $\deg f = [K(X) : K(Y)]$

We will define $f^*: \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(X)$ as follows

Let $q \in Y$, $t \in \mathcal{O}_{Y,q}$: uniformizer ie $(t) = m_q$ and $\nu_q(t) = 1$. $\longleftarrow$ local defining equation of q.

<u>Def</u> $f^* q := \sum_{f(p)=q} \nu_p(t)\, p$

$\rightsquigarrow$ linearly extend to $\mathrm{Div}(Y) \xrightarrow{f^*} \mathrm{Div}(X)$

<u>Prop</u> $\deg(f^* D) = \deg f \cdot \deg D$.

Pf) [Hartshorne, II. Prop 6.9] $\square$

$\Rightarrow$ Outside $q_1 \cdots q_r \in Y$, f is $\deg f : 1$ map.

<u>Cor</u> If $D \in \mathrm{Div}(X)$ is a principal divisor, then

$$\deg(D) = 0$$

Hence $\deg : A_0(X) \longrightarrow \mathbb{Z}$.

pf) $D = (f)$, $f \in K(X)$, It induces

$$F : X \longrightarrow \mathbb{P}^1$$

s.t $F^*(0 - \infty) = (f)$.

$\deg(f) = \deg F^*(0-\infty) = \deg F \cdot \deg(0-\infty) = 0$.

$\boxtimes$

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Let's do one example.

<u>Prop</u> Let $g \geq 1$, $p, q \in X$. Then

$$p \sim q \quad \text{iff} \quad p = q.$$

Pf) Proof by contradiction. Suppose $\exists\, p \neq q$ s.t $p \sim q$. ie $\mathcal{O}(p) \cong \mathcal{O}(q)$. Then

$$\mathbb{C}\langle s_p, s_q \rangle \subseteq H^0(\mathcal{O}(p)).$$

where s_p & s_q : linearly independent.

$$f : X \longrightarrow \mathbb{P}^1 \quad [s_p(z) : s_q(z)].$$

Then $\deg f = 1 \Rightarrow K(X) = \mathbb{C}(t)$

$\Rightarrow X \cong \mathbb{P}^1$ $\lightning$

§2. Abel–Jacobi theorem.

(2.1) Picard group

* X = curve / $\mathbb{C}$

Let

$$\mathrm{Pic}^0(X) := \ker\left(\mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}\right)$$
$$= \ker\left(H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})\right).$$

From the exponential sequence, we have

$$H^0(\mathcal{O}_X^*) \longrightarrow H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

we have $\mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X) / H^1(X, \mathbb{Z})$.

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The Abel-Jacobi theorem says that

$$K(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}_0(X) \xrightarrow{AJ} \mathrm{Jac}(X)$$

is exact. This is equivalent to say that :

(i)
$$\begin{tikzcd} \mathrm{Div}_0(X) \arrow[r] \arrow[d, "AJ"'] & \mathrm{Div}_0(X)/\sim \; = \mathrm{Pic}^0(X) \arrow[dl, "\overline{AJ}"] \\ \mathrm{Jac}(X) & \end{tikzcd}$$

(ii) $\overline{AJ}$ is <u>injective</u>.

Later we will see that $\overline{AJ}$ is surjective too.

(2.2) Picard group of elliptic curves

Let (E, p_0) be a pointed elliptic curve.

Let $u : E \to \mathrm{Pic}^0(E)$, $\quad p \mapsto \mathcal{O}(p - p_0)$

<u>Prop</u> $\quad u : (E, p_0) \longrightarrow \mathrm{Pic}^0(E)$ is an isomorphism

Pf) · Injectivity ✓

· Surjectivity : Let $\quad L \in \mathrm{Pic}^0(E)$. $L(p_0) = L \otimes \mathcal{O}(p_0)$.

$$h^0(L(p_0)) - h^1(L(p_0)) = \deg L(p_0) + 1 - g = 1.$$

By the Serre duality,

$$h^1(L(p_0)) = h^0(L^\vee(-p_0)) \overset{\deg < 0}{=} 0.$$

Choose a section $s \in H^0(L(p_0))$. Then

$$s^{-1}(0) = p \quad \text{for some } p \in E$$

and $L(p_0) \simeq \mathcal{O}(p)$. So $L \cong \mathcal{O}(p - p_0)$. ✓

To finish the proof, you need to construct the inverse morphism. (or argue directly). $\blacksquare$

(2.3) AJ for elliptic curves.

In the next lecture, we will see that

$$\mathrm{Im}(d\mathrm{Tr}) \subseteq \ker(AJ)$$

is simple. We will prove ($\supseteq$) when $X = E$.

If $AJ(D) = 0$, $\exists\, f$: meromorphic function, st

$$(f) = D.$$

□ Jacobi θ-function.

Let $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$, $\tau \in \mathbb{H}$, $E = \mathbb{C}/\Lambda$.

- If $f : \mathbb{C} \to \mathbb{C}$ holo. fcn, Λ-periodic, $f \equiv$ const.
- But we can weaken the condition slightly to get an interesting fcns.

<u>Def</u> $\theta(z;\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n^2 \tau + 2nz)}$. $z \in \mathbb{C}$

"Jacobi θ function"

<u>Check</u> $\theta(z;\tau)$ converges absolutely & uniformly on any compact subset $\subset \mathbb{C}$.

$\theta(z;\tau)$ satisfies the elliptic transformation rules:

- $\theta(z+1;\tau) = \theta(z;\tau)$

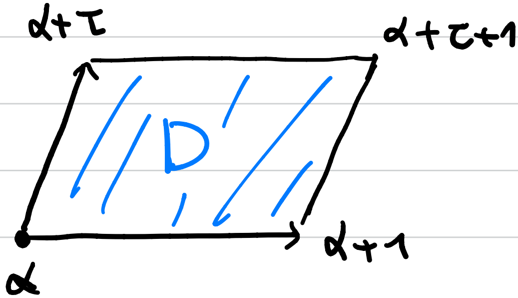
- $\theta(z+\tau;\tau) = \sum e^{\pi i(n^2\tau + 2n(z+\tau))}$

$$= \sum e^{\pi i((n+1)^2\tau + 2(n+1)z - \tau - 2z)}.$$

$$= \underbrace{e^{-\pi i(\tau+2z)}}_{\neq 0}\,\theta(z;\tau).$$

We can study the zero locus of $\theta(z;\tau)$ via the argument principle. + elliptic transform law.

Let $D \subset \mathbb{C}$: fundamental domain.



no zeros lie in ∂D.

CHECK :

$$(\#\text{ of zeros in } D) = \frac{1}{2\pi i}\int_{\partial D} \frac{\theta'(z;\tau)}{\theta(z;\tau)}\,dz$$

$$= 1$$

$$\text{zero locus of } \theta = \frac{1}{2\pi i}\int_{\partial D} \frac{z\theta'(z;\tau)}{\theta(z;\tau)}\,dz$$

$$= \frac{1+\tau}{2} + \alpha .$$

□ Proof of AJ theorem.

Let $p_1 \cdots p_r, q_1 \cdots q_r \in \mathbb{C}$ st

$$\sum p_j - \sum q_j = m + n\tau \quad \in \Lambda .$$

It is equivalent to say that corresponding divisor satisfies

$$AJ(\sum p_j - \sum q_j) = 0$$

Let

$$\varphi(z) = \frac{\prod_{i=1}^{r} \theta\left(z - p_j - \frac{1+\tau}{2}\right)}{\prod_{i=1}^{r} \theta\left(z - q_j - \frac{1+\tau}{2}\right)}$$

Lemma (i) $\varphi(z+1) = \varphi(z)$

(ii) $\varphi(z+\tau) = e^{2\pi i(\sum p_j - \sum q_j)} \varphi(z)$

$$= e^{2\pi i n\tau} \varphi(z). \qquad \square$$

Proof of $\ker(AJ) \subseteq \operatorname{Im}(\operatorname{div})$:

Let $\tilde{\varphi}(z) = e^{-2\pi i n z} \varphi(z)$. Then

- $\tilde{\varphi} \in K(E)$
- $\operatorname{div}(\tilde{\varphi}) = \sum p_j - \sum q_j$. $\blacksquare$