Lecture 4.

- Line bundles on V/Γ (Appell–Humbert)
- Positivity & Sections of L
- Lefschetz's theorem
- Projectivity of $J(C)$.

Many details are missing. You can find out details in Mumford's book

Recall

$X = V/\Gamma$: g-dim'l $\mathbb{C}$-torus

Goal Understand line bundles on X

$$
\begin{tikzcd}
 & F \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & E \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\{e_\gamma\} \in H^1(\Gamma, H^*) \arrow[r, "\delta"] \arrow[d, "\phi"', "\cong"] & H^2(\Gamma, \mathbb{Z}) \arrow[d, "\phi"', "\cong"] & \Lambda^2 H^1(\Gamma, \mathbb{Z}) \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
{[L]} \in H^1(X, \mathcal{O}_X^*) \arrow[r, "c_1"] & H^2(X, \mathbb{Z}) \arrow[d, phantom, "\rotatebox{90}{$\ni$}"] & \Lambda^2 H^1(X, \mathbb{Z}) \arrow[l, "\cong"] \\
 & c_1(L) = [E] &
\end{tikzcd}
$$

- $\gamma.(z, \lambda) = (z+\gamma, e_\gamma(z).\lambda)$ $\quad \Gamma \curvearrowright \mathbb{C} \times V$
- $e_\gamma = e^{2\pi i f_\gamma(z)}$,
- $F(\gamma_1, \gamma_2) = f_{\gamma_2}(\gamma_1 + z) - f_{\gamma_1+\gamma_2}(z) + f_{\gamma_1}(z) \in \mathbb{Z}$ $\quad (*)$
- $E(\gamma_1, \gamma_2) = F(\gamma_1, \gamma_2) - F(\gamma_2, \gamma_1) : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$ s.t. $\quad (**)$

after $\mathbb{R}$-linearly extended to $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$

$$E(ix, iy) = E(x, y) \quad \forall x, y \in V \qquad (1.1)$$

- $H(x,y) := E(ix, y) + iE(x, y)$: Hermitian form.

($\mathbb{C}$-linear on the first factor)

§1. Kernel of c_1

Suppose we are given an alternating form

$$E : \Gamma \times \Gamma \longrightarrow \mathbb{Z} \qquad \text{st}$$

E satisfies $\boxed{(1.1)}$. $\leftarrow$ Hodge structure of X

Q) Which $\{f_\gamma \in \mathcal{O}(V)\}_{\gamma \in \Gamma}$ satisfy $(*)$ & $(**)$?

Try $f_\gamma(z)$ which is affine in z.

$$f_\gamma(z) = \frac{1}{2i} H(z, \gamma) + \beta_\gamma \qquad \beta_\gamma \in \mathbb{C}.$$

<u>CHECK</u> $\{f_\gamma\}$ satisfies $(**)$.

$$F(\gamma_1, \gamma_2) = f_{\gamma_2}(\gamma_1 + z) - f_{\gamma_1+\gamma_2}(z) + f_{\gamma_1}(z) \in \mathbb{Z} \qquad (*)$$

$$\Leftrightarrow i\beta_{\gamma} + i\beta_{\gamma_2} - i\beta_{\gamma_1+\gamma_2} + \frac{1}{2} H(\gamma_1, \gamma_2) \in i\mathbb{Z}$$

Write $\quad i\beta_\gamma = b_\gamma + \frac{1}{4} H(\gamma, \gamma), \qquad b_\gamma \in \mathbb{C}.$

$$\Leftrightarrow \quad b_{\gamma_1} + b_{\gamma_2} - b_{\gamma_1+\gamma_2} + \frac{1}{2} i E(\gamma_1, \gamma_2) \in i\mathbb{Z}.$$

Let $l(z) \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Then replacing b_γ to $b_\gamma - l(\gamma)$ gives the same cocycle $\{e_\gamma\}$

$\Rightarrow$ May assume $\quad b_\gamma \in i\mathbb{R}$

Write $\alpha(\gamma) = e^{2\pi i b_\gamma}$. $\alpha : \Gamma \longrightarrow U(1) = \{z \in \mathbb{C} : |z| = 1\}$.

$$(*) \iff \frac{\alpha(\gamma_1+\gamma_2)}{\alpha(\gamma_1)\alpha(\gamma_2)} = \underbrace{e^{\pi i E(\gamma_1,\gamma_2)}}_{\in \{\pm 1\}} \qquad (*)'$$

"Semi-character"

<u>Check</u> For any E, $\exists\, \alpha : \Gamma \to U(1)$ satisfying $(*)'$.

<u>Lemma</u> H: Riemann form, $\alpha : \Gamma \to U(1)$ satisfying $(*)'$.

Then the action $\Gamma \circlearrowright V \times \mathbb{C}$ defined by

$$\gamma.(z.\lambda) = (z+\gamma, e_\gamma(z).\lambda), \qquad \text{where}$$

$e_\gamma(z) = \alpha(\gamma) e^{\pi H(z,\gamma) + \frac{1}{2}\pi H(\gamma,\gamma)}$ defines a line bundle

$L(H.\alpha) = V \times \mathbb{C} / \Gamma$ on V/Γ.

Moreover $c_1(L(H.\alpha)) = [E] \in H^2(X, \mathbb{Z})$.

<u>Check</u> $L(H_1,\alpha_1) \otimes L(H_2.\alpha_2) = L(H_1+H_2, \alpha_1\alpha_2)$.

<u>Thm</u> (Appell–Humbert)

Any line bundle on X is isomorphic to $L(H.\alpha)$ for uniquely determined H & α

Pf) [Mumford, p20-21].

§2. Sections of $L(H, \alpha)$

We start from discussing the positivity of a line bundle on X. (it will only depend on $c_1(L)$)

If H: Hermitian form,

$$H(x,x) = \overline{H(x,x)} \in \mathbb{R}. \quad \forall x \in V.$$

Def A Hermitian form H is <u>positive</u> <u>definite</u> if

$$H(x,x) > 0 \quad \forall x \in V.$$

$(\Leftrightarrow E(ix,x) > 0)$

Let Γ: rank $2g$ lattice.

Lemma $E: \Gamma \times \Gamma \to \mathbb{Z}$: nondegenerate alternating form.
Then $\exists\ d_1, \cdots d_g \in \mathbb{Z}_{>0}$ that satisfy $d_1 | \cdots | d_g$ and a basis $\{\gamma_1 \cdots \gamma_{2g}\}$ of Γ s.t

$$[E] = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}, \text{ where } \Delta = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_g \end{pmatrix}$$

Now we look at $H^0(X, L(H,\alpha))$.

$H^0(X, L(H,\alpha)) \overset{1:1}{\longleftrightarrow} H^0(V, V\times\mathbb{C}) = \mathcal{O}(V) \ni \theta$ satisfies transformation rule:

$$\theta(z+\gamma) = e_\gamma(z)\,\theta(z) = \alpha(\gamma)\, e^{\pi H(z,\gamma)+\frac{1}{2}\pi H(\gamma,\gamma)}\,\theta(z).$$

So we interchangeably use sections of $L(H,\alpha)$ and holomorphic functions on V with transformation property.

Example Let $\tau \in M_{g\times g}(\mathbb{C})$: $\tau = {}^t\tau$ and $\operatorname{Im}\tau > 0$

Let

$$\theta(z;\tau) = \sum_{m\in\mathbb{Z}^g} e^{2\pi i\left[\frac{1}{2}{}^tm\cdot\tau\cdot m + {}^tm z\right]}$$

transpose

Check θ defines a holomorphic function on $\mathbb{C}^g$ satisfies the transformation property for $\tau\mathbb{Z}^g \oplus \mathbb{Z}^g$

Prop If H is positive definite Riemann form satisfying (1.1), then

$$\dim H^0(L(H,\alpha)) = \sqrt{\det E}$$

Pf) BLACK BOX (Kodaira vanishing) If H : positive def then $H^p(L(H,\alpha)) = 0 \ \forall p > 0$.

$$\dim H^0(X, L(H,\alpha)) = \chi(L(H,\alpha))$$

$$= \int_X ch(L) \cdot td(T_X)$$

$$= \frac{1}{g!} \int_X c_1(L)^g .$$

It is possible to find a basis

$\{dx_1 \cdots dx_{2g}\}$ of $H^1(X, \mathbb{Z})$

s.t.

$$c_1(L) = \sum_{\alpha=1}^{g} d_\alpha \, dx_\alpha \wedge dx_{g+\alpha}$$

$$\rightsquigarrow c_1(L)^g = g! \prod_{\alpha=1}^{g} d_\alpha \, W_{vol} \in H^{2g}(X, \mathbb{Z}) \cong \mathbb{Z}$$

$\blacksquare$

There is a proof without using Kodaira vanishing in Mumford's book.

Eg if L: ample. $h^0(L^{\otimes r}) = r^g \cdot h^0(L)$, $r>0$

§3. Embedding to $\mathbb{P}^N$.

<u>Recall</u> $f: X \longrightarrow Y$ holo. map btw compact complex manifolds. Suppose

(i) f is injective

(ii) $df_x : T_xX \longrightarrow T_{f(x)}Y$ is injective for all $x \in X$.

Then $f(X) \subset Y$ is a $\mathbb{C}$-submanifold and

$X \xrightarrow{\cong} f(X)$. f is a closed immersion.

Let $\theta_0, \ldots, \theta_n \in H^0(X, L)$. without common zeros

$$\begin{array}{ccc} V & \xrightarrow{\tilde{u}} & \mathbb{C}^{n+1} \setminus \{0\} \\ \pi \downarrow & & \downarrow p \\ X & \xrightarrow{u} & \mathbb{P}^n \end{array} \qquad \tilde{u} = (\theta_0, \ldots, \theta_n)$$

<u>Lemma</u>. u is a closed immersion if

(i) u is injective

(ii) the matrix

$$\begin{pmatrix} \theta_0(z) & \frac{\partial \theta_0}{\partial z_1}(z) & \cdots & \frac{\partial \theta_0}{\partial z_g}(z) \\ \vdots & \vdots & & \vdots \\ \theta_n(z) & \frac{\partial \theta_n}{\partial z_1}(z) & & \frac{\partial \theta_n}{\partial z_g}(z) \end{pmatrix}$$ has rank $g+1$.

<u>Hint</u> $\ell \in \mathbb{C}^{n+1} - \{0\}$. $\ker(p) = \mathbb{C}\langle \ell \rangle$

The key observation is the following:
For $a \in X$, let

$$t_a : X \xrightarrow{\simeq} X \quad x \longmapsto x - a \qquad \text{"translation by } a\text{"}$$

<u>Lemma</u> $t_{a+b}^* L \otimes L \cong t_a^* L \otimes t_b^* L$ for all $a, b \in X$

Pf) By Appell-Humbert, we may assume $L = L(H, \alpha)$. The lemma follows from the equality

$$t_a^* L(H,\alpha) = L(H, \alpha e^{2\pi i \operatorname{Im} H(-,a)})$$

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In other words, if $\sum_{i=1}^r a_i = 0 \in V$, $\theta \in H^0(X, L)$,

$$\prod_{i=1}^r \theta(z + a_i) \in H^0(X, L^r).$$

<u>IDEA</u> If we take tensor products of positive def L, there is a choice of freedom to move around θ.

<u>Thm</u> (Lefschetz) If $c_1(L)$ is positive definite, then L^r defines a closed embedding to $\mathbb{P}^N$ for all $r \geq 3$

Pf) By RRoch calculation, $\exists$ nontrivial section of L.

Step 1 (No common zero) Let $z_0 \in V$ be a given point. Then $\exists$ $a \in V$ s.t

$$\theta(z_0 - a)\,\theta(z_0 + (r-1)a) \neq 0 .$$

Take $\theta(z-a)^{r-1}\,\theta(z+(r-1)a)$. By Lemma, it is a section of L^r. which does not vanish at z_0.

Step 2 (du is injective).

Let $\theta_0, \cdots, \theta_n$: basis of $H^0(L^r)$. Suppose $\exists$ $z_0 \in V$ s.t

$$\lambda_0 \theta_j(z_0) + \sum_{k=1}^{g} \lambda_k \frac{\partial \theta_j}{\partial z_k}(z_0) = 0 \quad \forall j = 0, \cdots, n$$

for some $(\lambda_0, \cdots \lambda_g) \in \mathbb{C}^{g+1} - \{0\}$. Choose $\Theta \in H^0(L)$ s.t

$$\Theta(z_0) \neq 0 .$$

For any $a, b \in V$, consider an entire function

$$\Theta_{ab}(z) = \theta(z-a)^{r-2}\,\theta(z-b)\,\theta(z+(r-2)a+b) \in H^0(L^r).$$

$$\rightsquigarrow \lambda_0 \theta_{ab}(z_0) + \sum_{k=1}^{g} \lambda_k \frac{\partial \theta_{ab}}{\partial z_k}(z_0) = 0$$

Let

$$\psi(z) = \sum_{k=1}^{g} \lambda_k \frac{\partial}{\partial z_k} \log \theta_k .$$

be a meromorphic function on V. Then,

$$(r-2)\psi(z_0-a) + \psi(z_0-b) + \psi(z_0+(r-2)a+b)$$

$$= \sum_{k=1}^{g} \lambda_k \frac{\partial}{\partial z_k} \log \theta_{ab}(z_0) = -\lambda_0 \qquad \forall a.b \in V.$$

For any a, $\exists\, b \in V$ s.t $\theta(z_0-b)\,\theta(z_0+(r-2)a+b) \neq 0$.
(ie. z_0-b & $z_0+(r-2)a+b$ is outside pole of $\psi(z)$)

$\Rightarrow \psi$ is a holomorphic function on V.
By transformation property of θ, we have

$$\psi(z+\gamma) = \psi(z) + \pi H(\lambda,\gamma) \quad (*) \; \forall \gamma \in \Gamma$$

where $H \leftarrow$ Hermitian form associated to $c_1(L)$

$\lambda = (\lambda_1, \cdots, \lambda_g) \in \mathbb{C}^g.$

$$\Rightarrow \frac{\partial}{\partial z_k}\left(\psi(z+\gamma) - \psi(z) \right) = 0 \qquad \forall\, k=1,\dots,g$$

$$\Rightarrow \frac{\partial}{\partial z_k}\psi(z) \text{ is constant} \qquad \forall\, k=1,\dots,g$$

so ψ is affine.

$$(\#) \Rightarrow \psi(\gamma) - \psi(0) = \pi H(\lambda, \gamma) \qquad \forall\, \gamma \in \Gamma$$

$\mathbb{R}$-linear

$$\Rightarrow \underbrace{\psi(z) - \psi(0)}_{\mathbb{C}\text{-linear}} = \underbrace{\pi H(\lambda, z)}_{\mathbb{C}\text{-antilinear}}$$

$$\Rightarrow H(\lambda, z) = 0 \qquad \forall\, z \in V$$

H: positive definite, in particular nondegenerate

$\Rightarrow \lambda = (\lambda_1, \dots, \lambda_g) = 0$ and $\lambda_0 = 0$. $\lightning$

<u>Step3</u> (Injectivity of u) Similar as in Step 2.

§ 4. Projective imbedding of $J(C)$

<u>Recall</u> : $J(C) = H^0(K_C)^\vee / \underbrace{H_1(C,\mathbb{Z})}_{"\Gamma}$.

Let $E : \Gamma \times \Gamma \to \mathbb{Z}$ be the intersection form

$$E(\gamma_1, \gamma_2) = [\gamma_1] . [\gamma_2]$$

In a.b cycles of $H_1(C,\mathbb{Z})$, E is nothing but

$$[E] = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$$

Let $\mathcal{B} = \{\gamma_1 \cdots \gamma_{2g}\} \in \Gamma$ be a basis . st

$$\Omega = (\gamma_1 \text{ --- } \gamma_{2g}) = (I_g \mid Z) \leftarrow \text{after taking normalized } \{\omega_i\}.$$

By Riemann's bilinear relation,

$$Z = {}^tZ \quad \text{and} \quad \operatorname{Im} Z > 0 .$$

In this normalized basis, we have

$$\gamma_1 = e_1 , \quad \text{- - -} \quad \gamma_g = e_g$$

where $e_k = (0, \cdots, \underset{\substack{\uparrow \\ k\text{ th}}}{1}, \text{ --- } 0) \in \mathbb{C}^g$.

Instead of $\mathcal{B}$, we can take a different basis

$$\mathcal{B}' = \{\gamma_1, \dots, \gamma_g, i\gamma_1, \dots, i\gamma_g\}$$

and the matrix for change of basis:

$$M = \begin{pmatrix} I_g & \mathrm{Re}\,Z \\ 0 & \mathrm{Im}\,Z \end{pmatrix}$$

For simplicity $R = \mathrm{Re}\,Z$, $S = \mathrm{Im}\,Z$.

If we write E wrt the basis $\mathcal{B}'$, we get

$$[E]_{\mathcal{B}'} = \begin{pmatrix} 0 & -S^{-1} \\ {}^tS^{-1} & {}^tS^{-1}(R - {}^tR)S^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -S^{-1} \\ S^{-1} & 0 \end{pmatrix}$$

Now it is easy to check that

$$E(ix, iy) = E(x, y) \quad \text{and} \qquad (1.1)$$

$$E(ix, x) > 0 \qquad \forall x \in \mathbb{C}^g \qquad \text{(positive def)}$$

assoc. to the intersection form

Rmk $h^0(L(H.\alpha)) = 1.$

The divisor Θ associated to $s \in H^0(L(H.\alpha))$ is called the theta divisor of $J(C)$.