Lecture 5.

- Summary on $\overline{\mathcal{M}}_{g,n}$
- Intersection theory
- ψ, κ classes
- Boundary strata

References:

- "ICM 2018 talk" by R. Pandharipande
- "Course on the moduli space of curves" by J. Schmitt

↳ links on our webpage

§1. Crash course on $\overline{\mathcal{M}}_{g,n}$.

<u>Def</u> $(C, p_1 \cdots p_n)$ is a <u>stable curve</u> of genus g if

(i) C is connected, complete, nodal curve of arithmetic genus g.

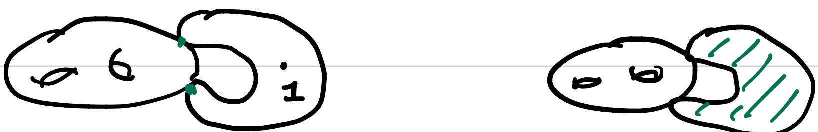
- nodal: $x \in C$ is either a smooth pt or analytic locally near x, $\mathcal{O}_C \simeq \mathbb{C}[u,v]/(uv)$. $\hat{\mathcal{O}}_{C,x} \cong \mathbb{C}[[u,v]]/(uv)$
- arithmetic genus $= \dim H^1(\mathcal{O}_C)$

(ii) $p_1, \cdots, p_n$: ordered n points on the smooth locus of C.

(iii) $|\mathrm{Aut}(C, p_1 \cdots p_n)| < \infty$.

- $\Leftrightarrow$ topological data of each irreducible component.

<u>Eg</u>



stable bc $g \geq 1$

stable bc $\#$ special pts ≥ 3

unstable component

$\overline{\mathcal{M}}_{g,n}(\mathbb{C}) = \{$ isomorphism class of stable curves $(C, p_1, \cdots p_n) \}$

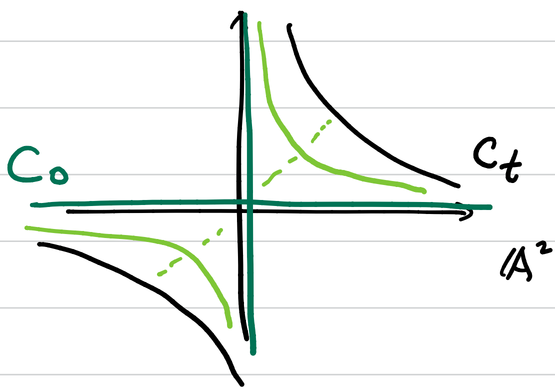
$\uparrow$ as a set.

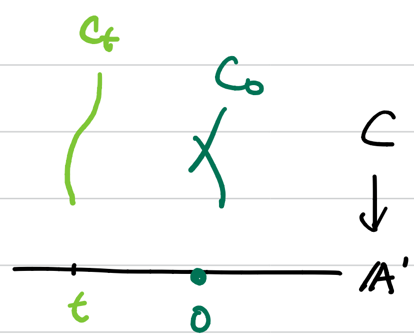
$\overline{\mathcal{M}}_{g,n}$ is the solution to a moduli problem.

Q) How to vary a stable curve in a family?

Example (affine curve)

Let $C = V(uv - t) \subset \mathbb{A}^3_{(u,v,t)}$
$$\downarrow$$
$$\mathbb{A}^1_t$$





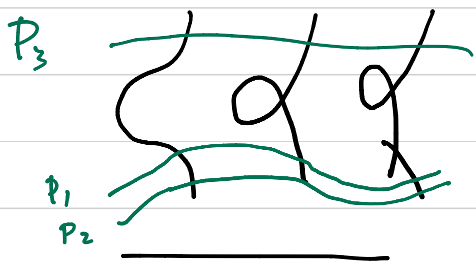
Key notion: Vary algebraic structure "continuously"
= flatness.

Let S : scheme / $\mathbb{C}$.

Def $\pi : C \to S$, $p_1, \cdots p_n : S \to C$ is a family of stable curves if

(i) π is a surjective, proper, flat morphism s.t any geometric fiber is a stable curve

(ii) $p_1, \cdots, p_n$: disjoint sections of π s.t image of p_i is in the smooth locus of π.



$$\begin{array}{c} C \\ \pi \downarrow \;\; \uparrow p_1, p_2, p_3 \\ S \end{array}$$

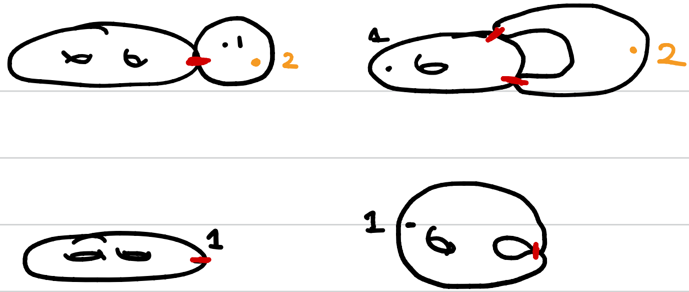
A relevant construction is the "forgetting morphism"

$$\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

$$(C, p_1, \cdots p_{n+1}) \longmapsto (\tilde{C}, p_1, \cdots p_n)$$

$\tilde{C} = C$ if $(C, p_1, \cdots p_n)$ is stable. otherwise we contract the unstable component.

$$\begin{tikzcd} \overline{M}_{2,2} \arrow[d] \\ \overline{M}_{2,1} \end{tikzcd}$$



Slogan: A family of a stable curve $C \xrightarrow{\pi} S$ corresponds to a morphism $S \to \overline{M}_{g,n}$ st

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & \overline{M}_{g,n+1} \arrow[d, "\pi"] \\ S \arrow[r] & \overline{M}_{g,n} \end{tikzcd}$$

Thm (Deligne-Mumford, Knutsen) Let $2g-2+n > 0$.
$\overline{M}_{g,n}$ is an irreducible, smooth, proper, DM-stack / $\mathbb{C}$ of dim $= 3g-3+n$.

space with group action.

§2. Intersection theory on $\overline{\mathcal{M}}_{g,n}$.

X, Y : topological spaces

∘ Singular (co)homology theory

- Functoriality : for any $f : X \to Y$

$$f_* : H_*(X) \longrightarrow H_*(Y)$$
$$f^* : H^*(Y) \longrightarrow H^*(Y)$$

- cup product

$$H^i(X) \otimes H^j(X) \longrightarrow H^{i+j}(X), \quad \alpha_1 \otimes \alpha_2 \mapsto \alpha_1 \cup \alpha_2$$

↳ ring structure.

- Cap product

$$H^i(X) \otimes H_d(X) \to H_{d-i}(X) \qquad \alpha \otimes \sigma \mapsto \alpha \cap \sigma$$

f^* & f_* are related by the projection formula

$$f_*(f^*\alpha \cap \sigma) = \alpha \cap f_* \sigma, \quad \alpha \in H^*(Y), \ \sigma \in H_*(X)$$

- Fundamental class If X is a compact, connected, oriented manifold,

$$H_{\dim_{\mathbb{R}} X}(X) \cong \mathbb{Q}\langle \underbrace{[X]}_{\text{fundamental class}} \rangle$$

st

$$\cap[X] : H^k(X) \xrightarrow{\cong} H_{\dim_{\mathbb{R}} X - k}(X)$$

"Poincare duality"

• Back to $\overline{\mathcal{M}}_{g,n}$

· $\overline{\mathcal{M}}_{g,n}$ has a coarse moduli space

$$\pi : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{M}_{g,n} \qquad \text{i.e.}$$

(i) $\overline{M}_{g,n}$ is a proper, irreducible scheme / $\mathbb{C}$ of $\dim_{\mathbb{C}} = 3g-3+n$

(ii) π induces a bijection of closed points.

(iii) π is proper.

⚠ Usually $\overline{M}_{g,n}$ is <u>not</u> smooth / $\mathbb{C}$.

We define

$\mathbb{C}$-analytic topology

$$H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) := H^*(\overline{M}_{g,n}, \mathbb{Q})$$
$$H_*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) := H_*(\overline{M}_{g,n}, \mathbb{Q})$$

Slogan (Co) Homology theory of $\overline{\mathcal{M}}_{g,n}$ behaves as if $\overline{\mathcal{M}}_{g,n}$ is a smooth compact $\mathbb{C}$-manifold.

<u>Reference</u> :

· Mumford, Towards an Enumerative Geometry of the Moduli Space of Curves.

• Degree map

<u>Def</u> Let X be a compact topological space.

$$\deg : H_0(X) \longrightarrow \mathbb{Q}$$

$$\sum a_i p_i \longmapsto \sum a_i$$

(isom if X is connected). Often we write $\int_X -$.

When X : Deligne-Mumford stack (such as $\overline{\mathcal{M}}_{g,n}$) we have to adjust the degree map.

space with a group action

<u>Ex</u> μ_n = cyclic group of order n. $\curvearrowright$ pt trivially.

$$\mathrm{Id} : * \underset{\deg = n}{\longrightarrow} [* / \mu_n] \underset{\deg = \frac{1}{n}}{\longrightarrow} *$$

bc it is univ. μ_n - torsor

For $[C, p_1 \cdots p_n] \in \overline{\mathcal{M}}_{g,n}$,

$$\deg [C, p_1 \cdots p_n] = \frac{1}{|\mathrm{Aut}(C, p_1 \cdots p_n)|}$$

Motivating Questions :

(1) What is $H^*(\overline{\mathcal{M}}_{g,n})$?

(2) Can we relate enumerative questions on the geometry of curves to computations in $H^*(\overline{\mathcal{M}}_{g,n})$?

(3) Can we understand "reasonable" subspace

$$RH^*(\overline{\mathcal{M}}_{g,n}) \subseteq H^*(\overline{\mathcal{M}}_{g,n})$$

so that we can do (1) & (2) ?

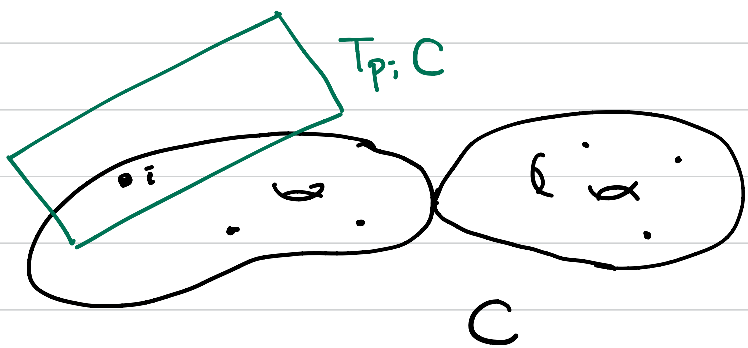
etc ...

Let's start from defining "natural classes" in $H^*(\overline{\mathcal{M}}_{g,n})$

§3. Line bundles on $\overline{\mathcal{M}}_{g,n}$ & ψ, κ-classes.

- There exist a line bundle called the "i-th cotangent line bundle" on $\overline{\mathcal{M}}_{g,n}$

$$\begin{array}{ccc} T^*_{p_i}C & \subset & \mathbb{L}_i \\ \downarrow & & \downarrow \\ [C, p_1 \cdots p_n] & \in & \overline{\mathcal{M}}_{g,n} \end{array} \qquad 1 \le i \le n$$



$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$ "psi-class"

- $\pi: X \to Y$: map between compact $\mathbb{C}$-manifolds.
  $r = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y$. Then

$$\begin{array}{ccc} H^i(X) & & H^{i-2r}(Y) \\ \text{PD} \downarrow \wr & & \downarrow \wr\ \text{PD} \\ H_{\dim_{\mathbb{R}} X - i}(X) & \xrightarrow{\pi_*} & H_{\dim_{\mathbb{R}} X - i}(Y) \end{array}$$

PD = Poincare duality

$\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$.

$\kappa_a := \pi_*(\psi_{n+1}^{a+1}) \in H^{2a}(\overline{\mathcal{M}}_{g,n})$

"Kappa class

In general we can multiply ψ & κ-classes:

<u>Eg.</u> $\psi_1^5 \psi_2^3 \kappa_1^2 \kappa_2 \in H^{24}(\overline{\mathcal{M}}_{15,2})$

$5 + 3 + 2 + 2$

$\shortparallel$

12

§4. Boundary strata

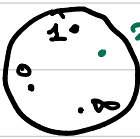
- "Boundary of $\overline{\mathcal{M}}_{g,n}$" $= \partial \overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \underbrace{\mathcal{M}_{g,n}}_{\text{locus where } (C,p_1 \ldots p_n) \text{ is smooth}}$

NOT a manifold with boundary.

Slogan: $\partial \overline{\mathcal{M}}_{g,n}$ has a <u>recursive</u> structure

(smaller genus or number of markings)

<u>Example</u> $\overline{M}_{0,3} = pt.$ $\quad (\mathbb{P}^1, x_1, x_2, x_3) \xrightarrow{PGL_2} (\mathbb{P}^1, 0, 1, \infty)$

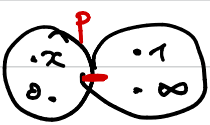
$\mathcal{M}_{0,4} =$ 

$= \mathbb{P}^1 \setminus \{0, 1, \infty\}$

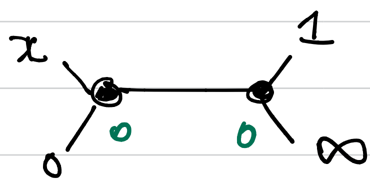
$\cap$

$\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$

$x \rightsquigarrow 0$



$\overline{M}_{0,\{0,x,p\}} \times \overline{M}_{0,\{p,1,\infty\}}$



<u>Check</u>: $H^*(\overline{\mathcal{M}}_{0,4}) = \mathbb{Q}[H]/(H^2), \quad H = [pt]^\vee$

$\Rightarrow \psi_1 = \psi_2 = \psi_3 = \psi_4 = H. \in H^2(\overline{\mathcal{M}}_{0,4})$

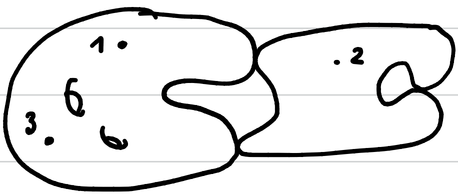
• Strata graph ($\Leftarrow$ dual graph of a stable curve)
↳ an organized way to describe $\partial\overline{M}_{g,n}$

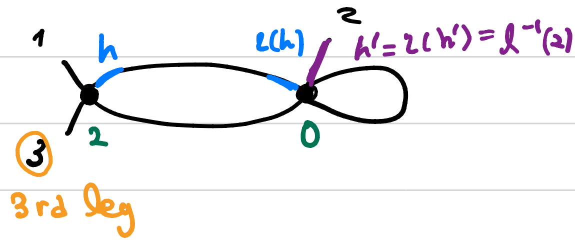
<u>Def</u> A stable graph of genus g with n markings is a data
$$\Gamma = (V, H, L, g: V \to \mathbb{Z}_{\geq 0}, v: H \to V, \iota: H \to H, \ell: L \xrightarrow{1:1} \{1, \dots, n\})$$
(i) $V \leftarrow$ set of vertices. $g(v) =$ "genus" at v
(ii) $H \leftarrow$ set of half-edges $v(h) =$ "vertex incident to h"
$n(v) = \#$ of incident half edges
(iii) $\iota: H \to H$ involution.
- if $\iota(h) = h \Rightarrow$ "leg" $L = \{h \in H \mid \iota(h) = h\}$
- if $\iota(h) = h'$, $h' \neq h \Rightarrow e = (h, h')$: edge.

s.t
(a) Γ is a connected graph
(b) $\forall v \in V$, $2g(v) - 2 + n(v) > 0$.

Vertex $\longleftrightarrow$ Irreducible component of C
Edge $\longleftrightarrow$ node of C.
Leg $\longleftrightarrow$ marked point of C

<u>Example</u>





$[C.p_1.p_2.p_3] \in \overline{M}_{4,3} \rightsquigarrow \Gamma_C$: "dual graph"

Let $M^{\Gamma} = \{(C, p_1 \cdots p_n) \mid \Gamma_C \cong \Gamma\} \subset \overline{M}_{g,n}$: irreducible locally closed nonempty subset of $\overline{M}_{g,n}$.

<u>Check</u> $\dim M^{\Gamma} = \dim \overline{M}_{g,n} - |E(\Gamma)|$

<u>Prop</u> Let Γ : stable graph of genus g, n markings.
Then there exist a morphism

$$\xi_{\Gamma} : \overline{M}_{\Gamma} := \prod_{v \in V(\Gamma)} \overline{M}_{g(v), n(v)} \longrightarrow \overline{M}_{g,n}$$

sending

$$\left(C_v, (q_h)_{h \mapsto v} \right)_{v \in V(\Gamma)} \longmapsto (C.p_1, \cdots p_n)$$

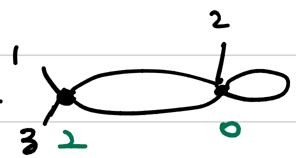
"gluing morphism"

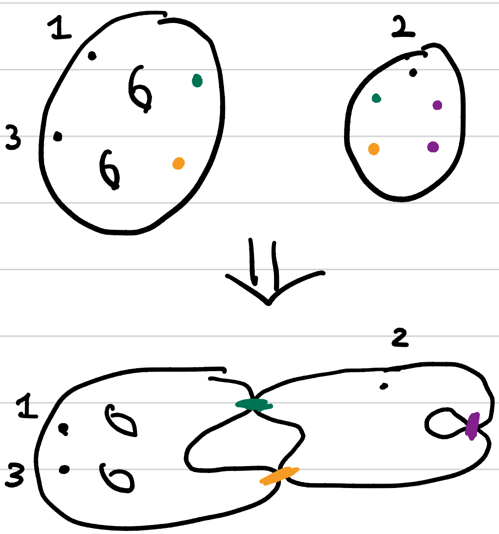
by gluing pairs (q_h, q_h') according to Γ.

Moreover,

(a) ξ_Γ is finite and

(b) $\xi_\Gamma(\overline{\mathcal{M}}_\Gamma) = \underbrace{\overline{\mathcal{M}^\Gamma}}_{\text{closure in } \overline{\mathcal{M}}_{g,n}.} \subset \overline{\mathcal{M}}_{g,n}$

<u>Example</u> $\Gamma =$ 



$$\begin{array}{c} \in \overline{\mathcal{M}}_{2,4} \times \overline{\mathcal{M}}_{0,5} \\ \big\downarrow \xi_\Gamma \\ \in \overline{\mathcal{M}}_{4,3} \end{array}$$

· Tautological Classes

Now we can combine ψ, κ classes & boundary classes to produce "natural classes" in $H^*(\overline{\mathcal{M}}_{g,n})$.

Γ : stable graph of genus g, n markings

$\alpha \in \prod_{v \in V(\Gamma)} H^*(\overline{\mathcal{M}}_{g(v),n(v)})$: product of ψ & κ-classes on each $\overline{\mathcal{M}}_{g(v),n(v)}$.

<u>Def</u> A tautological class associated to $[\Gamma, \alpha]$ is a class defined by

$$\xi_{\Gamma *}(\alpha) \in H^*(\overline{\mathcal{M}}_{g,n}).$$

Q) What is the product structure of $H^*(\overline{\mathcal{M}}_{g,n})$?

$$\xi_{\Gamma_1 *}(\alpha_1) \cup \xi_{\Gamma_2 *}(\alpha_2)$$

Is it still a linear combination of tautological classes?

$\longrightarrow$ Yes. Computer program. (Johannes will continue)