Lecture 7. ψ-class, String, Dilaton Eq.

- Definition of ψ-class
- Non-cartesian diagram
- String Equation
- Dilaton Equation.

⚠ We will see 3 different interpretations of ψ-classes.

§1. (Formal) Definition of ψ-classes

☐ Relative dualizing sheaf

Let X : nonsingular, projective variety / k of $\dim = n$.

$\Omega^1_{X/k}$ = sheaf of algebraic 1-forms on X

$T_{X/k} = (\Omega^1_{X/k})^\vee$ and it is a loc. free sheaf of rk $= n$.

$\omega_{X/k} = \Omega^n_{X/k} = \Lambda^n \Omega^1_{X/k}$: canonical line bundle

$\omega_{X/k}$ is the dualizing sheaf of X bc for any locally free sheaf $\mathcal{E}$,

$$H^i(X, \mathcal{E}) \cong H^{n-i}(X, \mathcal{E}^* \otimes \omega_{X/k})^\vee$$

"Serre duality"

- For a family of smooth curves $\pi : C \to S$, $\Omega^1_{C/S}$: line bundle on C.
- When π is not smooth, $\Omega_{C/S}$ is no longer an invertible sheaf.

Idea The dualizing sheaf $\omega_\pi = \omega_{C/S}$ is an invertible sheaf on C.

↳ The magic word is the "lci morphism".

Def A morphism $\pi : C \to S$ is called locally complete intersection if Zariski locally on C, $\exists$ factorization

$$\begin{tikzcd} C \arrow[r, hook, "\overset{I \swarrow}{i}"] \arrow[d, "\pi"'] & P \arrow[dl, "p"] \\ S & \end{tikzcd}$$

i : regular imbedding of codim $= r$

p : smooth

In particular the conormal sheaf I/I^2 of i is locally free. Then

$$\omega_{C/S} = i^*\omega_{P/S} \otimes \Lambda^r (I/I^2)^\vee$$

is well defined line bundle on C (ie independent of the choice of factorization). Moreover it is the dualizing sheaf. of π.

Fact $\pi : C \longrightarrow S$: family of nodal curves. Then π is lci.

When $S = \operatorname{Spec} k$, $p \in C^{sm}$. Then

$$\omega_{C,p} = \Omega^1_{C,p} = T^*_p C .$$

Def Let $\pi : \overline{M}_{g,n+1} \longrightarrow \overline{M}_{g,n}$ and $p_i : \overline{M}_{g,n} \longrightarrow \overline{M}_{g,n+1}$ be the i-th section. Then

$$\mathbb{L}_i = p_i^* \omega_\pi \qquad 1 \leq i \leq n .$$

Check If C/k is a nodal curve, $H^0(\omega_C) = H^1(\mathcal{O}_C)^\vee$.

$\Rightarrow$ $\deg(\omega_C) = 2g-2$ (by R.Roch)

Let's get a formula for ψ_i in terms of boundary strata.

We have $p_i : \overline{M}_{g,n} \simeq \overline{M}_{g,n} \times \overline{M}_{0,3} \longrightarrow \overline{M}_{g,n+1}$, defined by

Let $D_i = \operatorname{Im} p_i$. "i th section".

<u>Lemma</u> $\psi_i = -\pi_*[D_i]^2 \in H^2(\overline{M}_{g,n})$.

Pf)

$$\begin{tikzcd}
\overline{M}_{g,n} \arrow[r, "id"] \arrow[d, "id"'] \arrow[dr, phantom, "\ulcorner", very near start] & \overline{M}_{g,n} \arrow[d, "p_i"] & \\
\overline{M}_{g,n} \arrow[r, "p_i"] & \overline{M}_{g,n+1} \arrow[r, "\pi"] & \overline{M}_{g,n}
\end{tikzcd}$$

By the deformation theory argument, we saw that

$$\mathcal{N}_{p_i} \simeq \mathbb{L}_i^\vee$$

Now it follows from the excess intersection theory ☒

<u>Ex</u> Fill out the details of the proof.

§2. Non-cartesian diagram.

- Computation among tautological classes involves various pullbacks & pushforwards along π.

- In particular, it is very important to understand the following diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n\cup\{x,y\}} \arrow[dr, "G"'] \arrow[drr, bend left, "\rho_x"] \arrow[ddr, bend right, "\rho_y"'] & & \\
& \overline{\mathcal{M}}_{g,n\cup\{x\}} \times_{\overline{\mathcal{M}}} \overline{\mathcal{M}}_{g,n\cup\{y\}} \arrow[r, "\sigma_x"] \arrow[d, "\sigma_y"'] & \overline{\mathcal{M}}_{g,n\cup\{y\}} \arrow[d, "\pi_y"] \\
& \overline{\mathcal{M}}_{g,n\cup\{x\}} \arrow[r, "\pi_x"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd} \qquad (*)$$

Two issues :

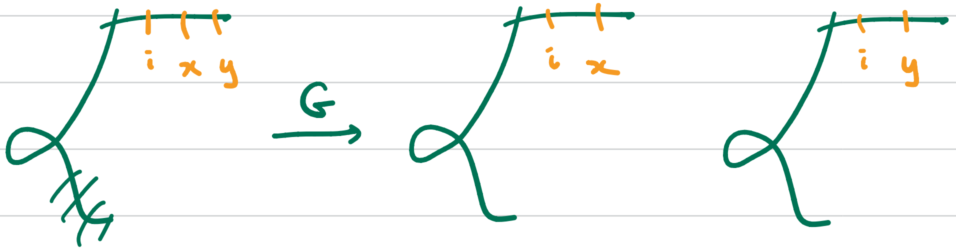
(i) The fiber product is <u>not</u> smooth.

→ This is fine because σ_x, π_x, are lci of relative dimension 1. The excess intersection formula

$$\pi_x^* \pi_{y*} = \sigma_{y*} \sigma_x^*$$

still holds

(ii) G is not an isomorphism.



→ At least G is birational, so

$$G_*[\overline{\mathcal{M}}_{g,n\cup\{x,y\}}] = [\overline{\mathcal{M}}_{g,n\cup\{x\}} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n\cup\{y\}}]$$

$$\Longrightarrow \quad \pi_x^* \pi_{y*} = \rho_{y*} \rho_x^*$$

So we think of the following diagram as if is a fiber diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n\cup\{x,y\}} \arrow[r, "\rho_x"] \arrow[d, "\rho_y"'] \arrow[dr, phantom, "\circlearrowright"] & \overline{\mathcal{M}}_{g,n\cup\{x\}} \arrow[d, "\pi_x"] \\ \overline{\mathcal{M}}_{g,n\cup\{y\}} \arrow[r, "\pi_y"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \qquad (*')$$

§ 3. String Equation.

$$\begin{tikzcd} \pi : \overline{M}_{g,n+1} \arrow[r] & \overline{M}_{g,n} \arrow[l, bend left, "p_i"] \end{tikzcd} \qquad D_i = p_i \overline{M}_{g,n}$$

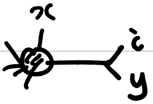
Lemma (1) $\pi^*\psi_i = \psi_i - [D_i]$

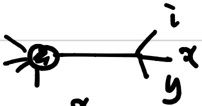
(2) $\psi_i^a = \pi^*\psi_i^a + \pi^*\psi_i^{a-1} \cdot [D_i] \qquad a \geq 1.$

Pf)

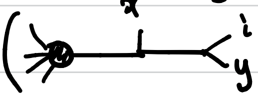
$$\begin{tikzcd} \overline{M}_{g,n\cup\{x,y\}} \arrow[r, "\rho_x"] \arrow[d, "\rho_y"'] & \overline{M}_{g,n\cup\{y\}} \arrow[d, "\pi_y"] \\ \overline{M}_{g,n\cup\{x\}} \arrow[r, "\pi_x"] & \overline{M}_{g,n} \end{tikzcd} \qquad (☆')$$

$\pi_x^*\left(-\pi_{y*}[D_i]^2\right) = -\rho_{y*}\rho_x^*[D_i]^2$

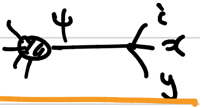
$= -\rho_{y*}\Big($ 

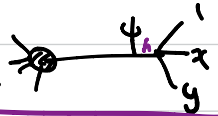
$+$ 

$\Big)^2$

$= \underbrace{-\rho_{y*}[D_i]^2}_{\psi_i} - 2\rho_{y*}\Big($ 

$\Big)$

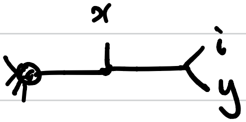
$+ \underbrace{\rho_{y*}\Big($ 

$\Big)}_{= 0}$ $+ \rho_{y*}\Big($ 

$\Big)$.

$\psi_1 =$ 

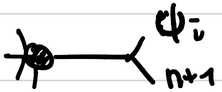
in $H^*(\overline{M}_{0,4})$

$= \psi_i - \rho_{y*}\Big($ 

$\Big).$

Check !

$$= \psi_i - [D_i]$$

(2) $\psi_i \cdot [D_i] = [D_i \cdot \psi_i] = [$ 

$] = 0$

Induction on a.

$$\psi_i^{a+1} = \psi_i^a \cdot \psi_i$$
$$= (\pi^*\psi_i^a + \pi^*\psi_i^{a-1}[D_i]) \cdot \psi_i$$
$$= \pi^*\psi_i^a(\pi^*\psi_i + [D_i])$$
$$= \pi^*\psi_i^{a+1} + \pi^*\psi_i^a[D_i]$$ ∎

Recall :

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}$$

exponent. not the i th marking.

For each $\sigma \in S_n$, $\exists$ isomorphism

$$\sigma : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

permuting $[n]$ - markings.

Ex Check $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_{\sigma(1)}^{k_1} \cdots \psi_{\sigma(n)}^{k_n}$.

String Eq. $$\langle \tau_0 \prod_{i=1}^{n} \tau_{k_i} \rangle_{g,n+1} = \sum_{k_i>0} \langle \tau_{k_i-1} \prod_{j\neq i} \tau_{k_j} \rangle_{g,n}$$

Proof) Since i-th section is disjoint from j-th section, $(i \neq j)$ we have

$$[D_i] . [D_j] = 0$$

$$\underbrace{\prod_{i=1}^{n} \psi_i^{k_i}}_{\text{on } \overline{\mathcal{M}}_{g,n+1}} = \prod_{i=1}^{n} \left(\pi^*\psi_i^{k_i} + \pi^*\psi_i^{k_i-1}[D_i]\right)$$

$$= \prod_{i=1}^{n} \pi^*\psi_i^{k_i} + \sum_{k_i>0} [D_i] \cdot \pi^*\left(\psi_i^{k_i-1} \prod_{j\neq i} \psi_j^{k_j}\right)$$

Now $\pi_*\pi^*\alpha = \pi_*(\pi^*\alpha . \mathbb{1}) = \alpha . \pi_*\mathbb{1} = 0$, so

$$= \sum_{k_i>0} \psi_i^{k_i-1} \psi_j^{k_j} . \underbrace{\pi_* D_j}_{\pi_* P_{j*}\mathbb{1} = \mathbb{1}} .$$

∎

Two consequences in $g=0$.

① The taut ring $RH^*(\overline{\mathcal{M}}_{0,n})$ is generated by boundary strata.

↳ $\psi_i =$  in $H^*(\overline{\mathcal{M}}_{0,4})$

$\rightsquigarrow$ use ψ - pullback formula $\Rightarrow$ $\psi_i = \sum$ boundary strata for all $n \geq 4$.

$\rightsquigarrow$ use boundary intersection formula $\Rightarrow$ all ψ - monomials are $\sum$ boundary

$\rightsquigarrow$ pushforward of boundary strata is boundary.

In fact, $RH^*(\overline{\mathcal{M}}_{0,n}) = H^*(\overline{\mathcal{M}}_{0,n})$.

<u>Ex</u> The coarse moduli space of $\overline{\mathcal{M}}_{1,1}$ is $\mathbb{P}^1$.

$\Rightarrow$ Show that $RH^*(\overline{\mathcal{M}}_{1,n})$ is also generated by boundary strata

② If $\sum k_i = n-3$, $\langle \prod \tau_{k_i} \rangle_0 = \dfrac{(n-3)!}{\prod k_i!}$

$\Rightarrow$ Use $\langle \tau_1 \rangle_{0,4} = 1$ & Induction on n.

§4. Dilaton Equation.

Notation: For a family of n-pointed nodal curve.

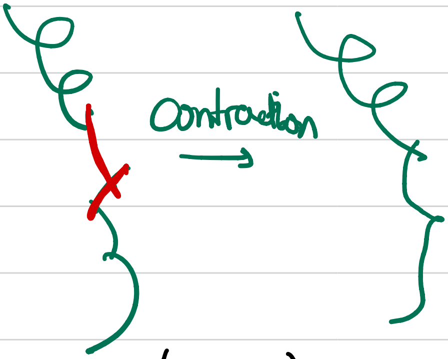
$$X \xrightarrow{\pi} S \ , \ p_1, \cdots p_n : S \to X$$

write

$$K_\pi := c_1(\omega_\pi(p_1+\cdots+p_n)) \quad \in H^2(X).$$

<u>Fact</u> [Knudsen, '83] Let $(\pi: X \to S. p_1 \cdots p_n)$ be a family of semi-stable nodal curves (ie $2g(v)-2+n(v) \geq 0$) Then $\exists!$ $(\pi': X' \to S. p_1 \cdots p_n)$ and $f: X \to X'$ which contracts unstable components

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$



Moreover $f^* \omega_{X'/S}(\Sigma p_i) \simeq \omega_{X/S}(\Sigma p_i)$

$$\begin{tikzcd}
\overset{K_{\rho_y}}{\overline{\mathcal{M}}_{g,n\cup\{x,y\}}} \arrow[r, "\rho_y"] & \overset{K_{\pi_x}}{\overline{\mathcal{M}}_{g,n\cup\{x\}}} \arrow[r, "\pi_x"] \arrow[l, bend left, "\tau_x"] & \overline{\mathcal{M}}_{g,n} \\
\cup & \psi_x &
\end{tikzcd}$$

$D_{x,y} = [$  $]$

By [Knudsen, '83], $\rho_y^* K_{\pi_x} = K_{\rho_y} - [D_{xy}]$. $(*)$

Goal. $\psi_x = K_{\pi_x}$ in $H^2(\overline{\mathcal{M}}_{g,n\cup\{x\}})$.

Lemma $\rho_{y*}(\gamma \cdot [D_{xy}]) = \tau_x^* \gamma \quad \forall\, \gamma \in H^*(\overline{\mathcal{M}}_{g,n\cup\{x,y\}})$

pf)
$$\begin{aligned}
\rho_{y*}(\gamma \cdot [D_{xy}]) &= \rho_{y*}(\gamma \cdot \tau_{x*}\mathbb{1}) \\
&= \rho_{y*}(\tau_{x*}(\tau_x^*\gamma \cdot \mathbb{1})) \\
&= \tau_x^*\gamma
\end{aligned}$$
☒

Proof of Goal)

$$\begin{aligned}
\psi_x &= \tau_x^*(c_1(\omega_{\rho_y})) \\
&= \tau_x^*(c_1(\omega_{\rho_y}) + [D_1] + \cdots + [D_n] + [D_{xy}] - [D_{xy}])
\end{aligned}$$

↖ bc τ_x does not hit $D_1, \text{--} D_n$

$$= \tau_x^*(K_{\rho_y} - [D_{xy}])$$

$$= \rho_{y*}\Big((K_{\rho_y} - [D_{xy}]) \cdot [D_{xy}] \Big)$$

$\uparrow$ lemma

$$= \rho_{y*}(\rho_y^* K_{\pi_x} \cdot [D_{xy}])$$

$\uparrow$ Knudsen

$$= K_{\pi_x}$$

$\uparrow$ projection formula. $\square$

Cor $K_0 = \pi_*(\psi_{n+1}) = \deg(\omega_\pi(D))\, \mathbb{1}.$

$$= (2g-2+n)\, \mathbb{1}$$

Dilaton Eq $\langle \tau_1 \prod_{i=1}^{n} \tau_{k_i} \rangle_{g,n+1} = (2g-2+n) \langle \prod_{i=1}^{n} \tau_{k_i} \rangle_{g,n}$.

proof) $\pi_*\Big(\psi_{n+1} \prod_{i=1}^{n} \psi_i^{k_i}\Big) = \pi_*\Big(\psi_{n+1} \prod_{i=1}^{n} (\pi^*\psi_i^{k_i} + \pi^*\psi_i^{k_i-1}[D_i])\Big)$

$$= \pi_*\Big(\psi_{n+1} \prod_{i=1}^{n} \pi^*\psi_i^{k_i}\Big)$$

$\uparrow$ $\psi_{n+1} \cdot [D_i] = 0$

$$= \pi_*(\psi_{n+1}) \prod_{i=1}^{n} \psi_i^{k_i}$$

$$= (2g-2+n) \prod_{i=1}^{n} \psi_i^{k_i} \qquad \square$$

Ex Compute all $\langle \prod \tau_{k_i} \rangle_1$ in terms of $\langle \tau_1 \rangle_1 = \frac{1}{24}$.