Lecture 9. Introduction to $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

- Def of $\overline{\mathcal{M}}_{g,n}(X,\beta)$
- Boundary strata Def / Obs
- Examples : $X = \mathbb{P}^1$.

§1. Defn of the moduli space of stable maps.

X = nonsingular, projective variety $/ \mathbb{C}$.

$\beta \in H_2(X, \mathbb{Z}) \leftarrow$ singular homology.

$$\overline{M}_{g,n}(X,\beta)(\mathbb{C}) = \left\{ f : (C, p_1 \cdots p_n) \to X \;\middle|\; \begin{array}{l} (C, p_1 \cdots p_n) : \text{connected nodal curve} \\ p_a(C) = g, \; f_*[C] = \beta \\ |\mathrm{Aut}(f)| < \infty \end{array} \right\}$$

- Stability condition: $\mathrm{Aut}(f) \subset \mathrm{Aut}(C, p_1 \cdots p_n)$ s.t

$$\begin{tikzcd}
(C, p_i) \arrow[r, "f"] & X \\
(C, p_i) \arrow[u, "g"] \arrow[ur, "f"'] \arrow[ur, phantom, shift left=3, "\circlearrowleft"] &
\end{tikzcd}$$

<u>Ex</u> $f : \mathbb{P}^1 \to \mathbb{P}^N$ which is not const. $\Rightarrow [f]$ has only finitely many automorphisms.

$\rightsquigarrow$ $(C, p_1 \cdots p_n)$ can contain unstable components.

<u>Thm</u> (Kontsevich) $\overline{M}_{g,n}(X,\beta)$ is a proper Deligne-Mumford stack $/ \mathbb{C}$.

□ Canonical maps.

There are canonical maps associated to $\overline{\mathcal{M}}_{g,n}(X,\beta)$:

$\mathfrak{M}_{g,n}$ = moduli space of nodal curves of genus g, n markings (no stability) ← locally finite type algebraic stack

universal curve

$$\begin{tikzcd} \mathcal{C} \cong \overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[d, "\pi"'] & \\ {[f]} \in \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[r, "ev_i"] \arrow[d, "p"] & X \\ \mathfrak{M}_{g,n} & \end{tikzcd}$$

$ev_i([f]) = f(p_i)$

$p([f]) = (C, p_1 \cdots p_n)$

$1 \le i \le n$

$\mathcal{M}_{g,n}(X,\beta) \subset \overline{\mathcal{M}}_{g,n}(X,\beta)$ is the locus where C is smooth.

When $2g-2+n>0$, we know:

$\overline{M}_{g,n}$: smooth, irreducible, DM stack of dim $= 3g-3+n$

$\cup$

$M_{g,n}$ $\leftarrow$ nonempty open substack.

Unlike $\overline{M}_{g,n}$, $\overline{M}_{g,n}(X,\beta)$ can be as complicated as possible!

a) can be empty

b) can be disconnected

c) can have many irred components w/ different dim

d) can have arbitrarily worse singularity. (due to Vakil)

Examples (i) X = elliptic curve $\Rightarrow \overline{M}_{0,n}(X,\beta) = \phi$ if $\beta \neq 0$.

(ii) $X \subset \mathbb{P}^4$ general deg 5 hypersurface

$\Rightarrow \exists$ finitely many (2875) lines $\subset X$

$\Rightarrow \overline{M}_{0,0}(X,1)$ is a disjoint union of finitely many pts.

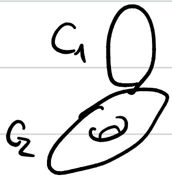
Despite all those difficulties, we can understand meaningful geometry of $\overline{M}_{g,n}(X,\beta)$! ('92~)

§2. Boundary strata

Recall: We described the boundary of $\overline{\mathcal{M}}_{g,n}$ in terms of stable graphs

$\rightsquigarrow$ play similar games with decorated stable graphs.

Ex f :

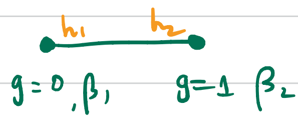


$\longrightarrow X$

$f_*[C_1] = \beta_1$

$f_*[C_2] = \beta_2$

$\rightsquigarrow \Gamma =$



- Gluing homomorphism. $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma(X) \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)$.

Ex Let's try:

$\overline{\mathcal{M}}_{0,1}(X,\beta_1) \times \overline{\mathcal{M}}_{1,1}(X,\beta_2)$ $\qquad$ $\overline{\mathcal{M}}_1(X,\beta)$

The image of two legs should map to the same point!

$$\begin{tikzcd}
& \overline{\mathcal{M}}_\Gamma(X) \arrow[dl, "\xi_\Gamma"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \overline{\mathcal{M}}_{1,1}(X,\beta_2) \arrow[d, "ev_{h_2}"] \\
\overline{\mathcal{M}}_{g,n}(X,\beta) & \overline{\mathcal{M}}_{0,1}(X,\beta_1) \arrow[r, "ev_{h_1}"'] & X
\end{tikzcd}$$

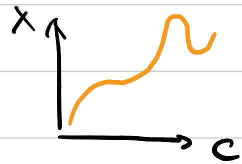
ξ_Γ is a finite morphism. We will see that the image of ξ_Γ is not an actual divisor of $\overline{\mathcal{M}}_{g,n}(X,\beta)$!

▫ Deformation & Obstruction.

↳ We will study this carefully later!

Suppose we want to vary $f: C \to X$ in a family.

(i) deform C $\rightsquigarrow$ $\dim = 3g-3+n$



(ii) fix C & vary f.

$\sim$ locally imbed into $\mathrm{Hilb}(C \times X)$.

$[f] \longleftrightarrow$ $C \hookrightarrow C \times X$ by graph of f.

$$\mathrm{Def}(f) = H^0(C, N_{C/C\times X}) = H^0(C, f^*TX)$$
$$\mathrm{Obs}(f) = H^1(C, N_{C/C\times X}) = H^1(C, f^*TX).$$

$$\mathrm{vdim}_{[f]} := \chi(C, f^*TX) + 3g-3+n$$
$$= (1-g)(\dim X - 3) + \int_\beta c_1(f^*TX) + n$$

↑ R.Roch   independent of $[f]$!

__Fact__ vdim $\leq$ dimension of each irred component of $\overline{M}_{g,n}(X,\beta)$.

§3. Examples: $X = \mathbb{P}^1$

$\beta = d[\mathbb{P}^1] \in H_2(\mathbb{P}^1, \mathbb{Z})$, $d \geq 0$. $T\mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(2)$.

(A) $d=0$, $2g-2+n>0$

$[f]$ should be constant $\Rightarrow$ $\overline{M}_{g,n}(\mathbb{P}^1, 0) \cong \overline{M}_{g,n} \times \mathbb{P}^1$.

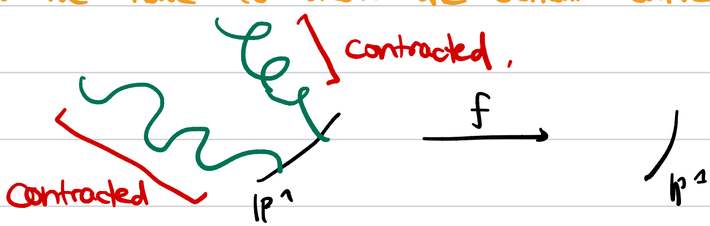
$$\text{vdim} = (1-g)(1-3)+n = \underbrace{2g-2+n \leq \dim}_{\text{difference} = g} = 3g-3+n$$

We will see the meaning of the difference later!

(B) $d=1$, $g \geq 1$

$M_g(\mathbb{P}^1, 1) = \emptyset$ because there is no degree 1 map $C \longrightarrow \mathbb{P}^1$, $g(C) \geq 1$.

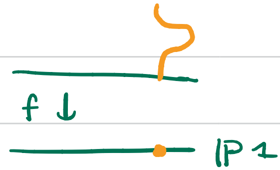
$\rightsquigarrow$ So we have to break the domain curve C



$\rightsquigarrow$ $\overline{M}_\Gamma(\mathbb{P}^1, 1)$ are not actual boundaries (codim $=1$)

$d = 1,\ g = 1.$ $\rightsquigarrow$ $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^1,1) \simeq \overline{\mathcal{M}}_\Gamma(\mathbb{P}^1)$ where

$\Gamma =$ 



__Check__ Let $C =$ rat'l nodal curve. Then $\nexists\ C \longrightarrow \mathbb{P}^1$ of $\deg = 1$

Last time: $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,1) = pt$. $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1,1) \cong \mathbb{P}^1$.

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1,1) \times_{\mathbb{P}^1} \overline{\mathcal{M}}_{1,1}(\mathbb{P}^1,0)$$
$$= \overline{\mathcal{M}}_{1,1} \times \mathbb{P}^1.$$

$$\begin{tikzcd}
 & \overline{\mathcal{M}}_{1,1} \times \mathbb{P}^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathcal{M}}_{1,1} \times \mathbb{P}^1 \approx \overline{\mathcal{M}}_{1,1}(\mathbb{P}^1,0) \arrow[d, "pr_2"] \\
\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1,1) = & \mathbb{P}^1 \arrow[r, "Id"] & \mathbb{P}^1
\end{tikzcd}$$

$\Rightarrow \text{vdim} = 2 = \dim \overline{\mathcal{M}}_1(\mathbb{P}^1,1).$

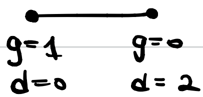
(c) $d=2$. $g=1$. $\overline{\mathcal{M}}_1(\mathbb{P}^1, 2)$

$$\text{vdim} = (1-g)(\dim X - 3) + 2d + n = 0 + 2\cdot 2 + 0 = 4.$$

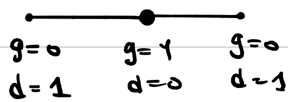
There are at most 3 irreducible components

(i) $\overline{\mathcal{M}}^{\text{main}}$: closure of $\mathcal{M}_{1,0}(\mathbb{P}^1, 2) \subset \overline{\mathcal{M}}_{1,0}(\mathbb{P}^1, 2)$.

(ii) $\overline{\mathcal{M}}_{\Gamma_1} = \overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1)$: $\Gamma_1 =$



(iii) $\overline{\mathcal{M}}_{\Gamma_2} = \overline{\mathcal{M}}_{\Gamma_2}(\mathbb{P}^1)$: $\Gamma_2 =$



smooth elliptic curve

(i) is the most interesting part. Let $f : C \longrightarrow \mathbb{P}^1$

By Riemann-Hurwitz,

$$\text{br}(f) = \deg(K_C) - 2\deg(K_{\mathbb{P}^1}) = 0 - 2(-2) = 4.$$

We have a map

$$\text{br} : \mathcal{M}_1(\mathbb{P}^1, 2) \longrightarrow \text{Sym}^4(\mathbb{P}^1) \cong \mathbb{P}^4.$$

<u>ASIDE</u> (stack structure) $C \subset \mathbb{P}^2$, $C = V(G)$. where

$G = y^2 - x^3 - ax^2 - bx - c$.

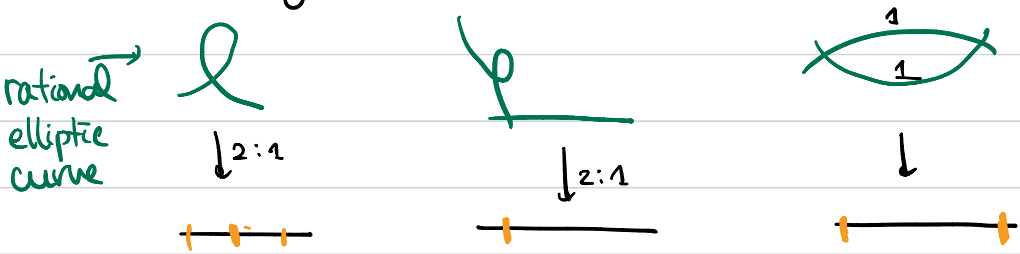
$$f : C \longrightarrow \mathbb{P}^1 \qquad (x, y) \longmapsto x$$

$\exists\ \sigma \in \mathrm{Aut}(C)$. $\quad \sigma(x,y) = (x,-y)$.

$$\begin{tikzcd} C \arrow[rr, "\sigma", "\sim"'] \arrow[dr, "f"'] & & C \arrow[dl, "f"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

$\Rightarrow$ $\mathrm{Aut}(f)$ is nontrivial for all $[f] \in \mathcal{M}_1(\mathbb{P}^1, 2)$

In the closure of $\mathcal{M}_{1,0}(\mathbb{P}^1, 2)$, we might have following configurations



It is not so obvious when $[f] \in \overline{\mathcal{M}}^{\text{main}}$.

<u>Thm</u> (Vakil) Let $g=1$. Then $[f] \in \overline{\mathcal{M}}^{\text{main}}$ if and only if

(i) f contracts no $g=1$ curve OR

(ii) let E = maximal connected $g=1$ contracted by f & E meets $C' := \overline{C \setminus E}$ at points $p_1 \cdots p_m$. Then

$$\{ df(T_{p_1} C'_1), \dots, df(T_{p_m} C') \}$$

is a linearly dependent set of $T_{f(E)} \mathbb{P}^1$.

Let's go back to our case:

$$\overline{M}_{\Gamma_1} = \overline{M}_{0,1}(\mathbb{P}^1,2) \times_{\mathbb{P}^1} \overline{M}_{1,1}(\mathbb{P}^1,0)$$

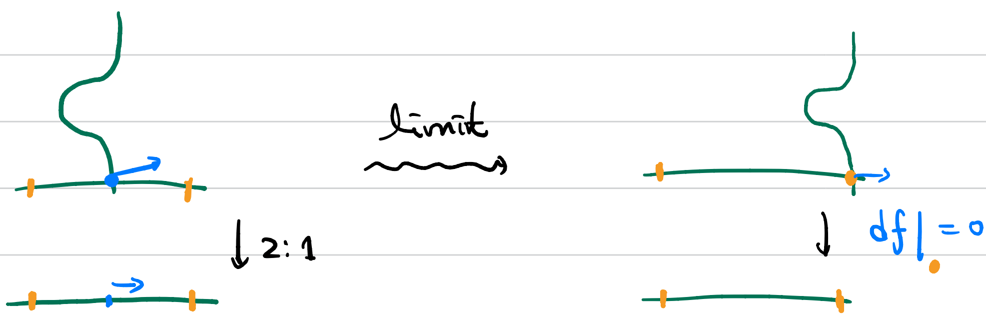
$$= \overline{M}_{0,1}(\mathbb{P}^1,2) \times \overline{M}_{1,1}$$

We saw $\overline{M}_{0,0}(\mathbb{P}^1,2) \cong \mathbb{P}^2$ (as a coarse moduli space)

$\Rightarrow \dim \overline{M}_{0,1}(\mathbb{P}^1,2) = 3$.

& $\dim \overline{M}_{\Gamma_1} = 4$ $\leftarrow$ dimension of "boundary" does not drop!

* General point of $\overline{M}_{\Gamma_1}$:



Not inside $\overline{M}^{main}$ Inside $\overline{M}^{main}$.

$$\overline{\mathcal{M}}_{\Gamma_2} = \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1,1) \times_{\mathbb{P}^1} \overline{\mathcal{M}}_{1,2}(\mathbb{P}^1,0) \times_{\mathbb{P}^1} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1,1)$$

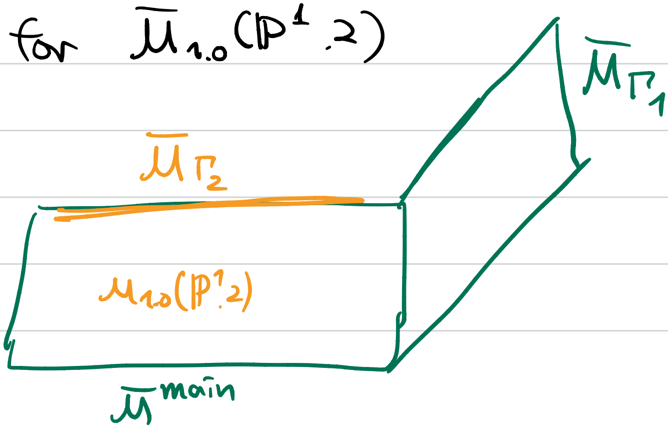
$$= \mathbb{P}^1 \times \overline{\mathcal{M}}_{1,2}$$

$\dim \overline{\mathcal{M}}_{\Gamma_2} = 3$.

General points in $\overline{\mathcal{M}}_{\Gamma_2}$

$\Rightarrow \overline{\mathcal{M}}_{\Gamma_2} \subset \overline{\mathcal{M}}^{main}$

$\rightsquigarrow$ Picture for $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^1,2)$



(b) $d = 2,\ g = 2$

* Let's start from $\mathcal{M}_{2,0}(\mathbb{P}^1, 2)$. $f : C \xrightarrow{2:1} \mathbb{P}^1$

By Riemann-Hurwitz,

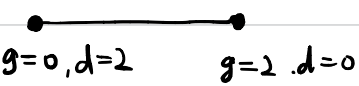
$$br(f) = \deg(K_C) - 2\deg(K_{\mathbb{P}^1})$$
$$= 2 - 2(-2) = 6.$$

$\Rightarrow$ $\mathcal{M}_{2,0}(\mathbb{P}^1, 6) \longrightarrow \mathbb{P}^6$.

$\Rightarrow$ $\dim \overline{\mathcal{M}}^{main} = 6$

vdim $= (1-g)(\dim X - 3) + 2d + n = -1(-2) + 2 \cdot 2 = 6$

* Consider a graph

$\Gamma =$ 

$\Rightarrow$ $\overline{\mathcal{M}}_\Gamma(\mathbb{P}^1) = \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, 2) \times_{\mathbb{P}^1} \overline{\mathcal{M}}_{2,1}(\mathbb{P}^1, 0)$

$\dim \overline{\mathcal{M}}_\Gamma(\mathbb{P}^1) = 2 + 1 + 3 + 1 + 1 - 1 = 7$

dimension of the "boundary stratum" jumps!

So irreducible components of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ can have different dimensions