Curves, Jacobians, and Modern Abel–Jacobi Theory

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Autumn Semester 2020

Our base field is $\mathbb{C}$.

1. Let $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ be an elliptic curve.
(a) Construct a meromorphic function on E with a pole of order exactly 3 at the origin.
(b) Is it the unique meromorphic function (up to scale) with this property?

2. Show that there is a non-algebraic complex torus.

3. Calculate the genus of a nonsingular degree d curve in $\mathbb{P}^2$.

4. Let X and Y be complex (compact, connected) Riemann surfaces of genus g_X and g_Y respectively. Suppose g_Y is strictly greater than g_X. Does there exist a nonconstant holomorphic map from X to Y? Answer with proof.

5. Let X be a complex (compact, connected) Riemann surface of genus greater than equal to one. Fix a point $p \in X$. Let $f_p \colon X \to \mathrm{Jac}(X)$ be a map defined by

$$f_p(q) = \mathsf{AJ}(p - q)\,, \quad q \in X\,.$$

(a) Prove f_p is holomorphic and injective.
(b) Is f_p an immersion (i.e. is the differential always injective)?

6. Let X and E be complex (compact, connected) Riemann surfaces of genus 2 and 1 respectively. Let $f \colon X \to E$ be a non-constant holomorphic map with $\deg(f) = 2$. Given a line bundle L of degree 0 on E, the pull-back $f^*(L)$ is a line bundle on X of degree 0.
(a) Construct the holomorphic map of Jacobians

$$\mathrm{Jac}(E) \to \mathrm{Jac}(X)$$

associated to the above pull-back.
(b) Does every genus 2 Riemann surface admit such a degree 2 map to some genus 1 Riemann surface?

7. An algebraic variety V is called *unirational* if there exists a dominant rational map $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow V$.
(a) Prove that the moduli space $\overline{M}_3$ is unirational.
(b) Is $\overline{M}_{3,1}$ also unirational?

8. Compute the (topological) Euler characteristic of $\overline{\mathcal{M}}_{0,5}$.

9. Write down all isomorphism classes of stable graphs with exactly two edges for $g = 3, n = 1$. Specify the order of automorphism of each stable graph.

10. Let

$$\xi \colon \overline{\mathcal{M}}_{2,2} \to \overline{\mathcal{M}}_{3,0}$$

be the morphism induced by gluing two markings. Let $D = \xi_*[\overline{\mathcal{M}}_{2,2}]$ be the divisor class in $H^2(\overline{\mathcal{M}}_{3,0})$. Compute $D \cdot D$ in terms of tautological classes.

11. Let $n \geq 3$. Prove that the tautological ring of $\overline{\mathcal{M}}_{0,n}$ is additively generated by boundary strata (i.e. without ψ or κ classes).

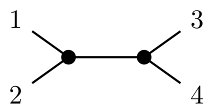
12. (a) Let $n \geq 3$ and let $a_1, \ldots, a_n$ be nonnegative integers with $\sum_i a_i = n-3$. Show that

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \frac{(n-3)!}{a_1! \cdots a_n!} \,.$$

(b) Calculate the following descendent integrals on $\overline{\mathcal{M}}_{1,3}$:

$$\langle \tau_0 \tau_0 \tau_3 \rangle_1 \,, \langle \tau_0 \tau_1 \tau_2 \rangle_1 \,, \langle \tau_1 \tau_1 \tau_1 \rangle_1 \,.$$

13. Let $\pi \colon \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{0,4}$ be the morphism induced by forgetting the 5-th and 6-th markings. Let

$D =$ 

be a boundary stratum in $H^2(\overline{\mathcal{M}}_{0,4})$. Compute $\pi^* D$ in $H^2(\overline{\mathcal{M}}_{0,6})$. You have to justify multiplicities of boundary strata.

14. Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)$ be the moduli space of stable maps to $\mathbb{P}^m$ of genus g with n markings.
(a) Describe the $\mathbb{C}$-points of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)$?
(b) Compute the virtual dimension of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)$ in terms of g, n, m and d.
(c) Let $g = 0$ and $d \geq 1$. For each

$$[f \colon (C, p_1, \ldots p_n) \to \mathbb{P}^m] \in \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)(\mathbb{C}) \,,$$

prove that $H^1(C, f^* T\mathbb{P}^m) = 0$. This result shows that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ is a smooth Deligne–Mumford stack.

15. (a) Show that $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 1)$ is birational to $\mathbb{P}^2 \times \mathbb{P}^2$.
(b) From (a), show that

$$N_1 := \langle \tau_0(\mathsf{p}) \tau_0(\mathsf{p}) \rangle_{0,1}^{\mathbb{P}^2} = 1 \,,$$

where $\mathsf{p} \in H^4(\mathbb{P}^2)$ is the point class.

16. The one dimensional torus $\mathbb{C}^*$ acts on $\mathbb{P}^1$ by

$$t.[z_0 : z_1] = [z_0 : t z_1] \,.$$

Describe all connected components of $\mathbb{C}^*$-fixed loci of $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, 3)$.

17. Let $T = (\mathbb{C}^*)^2$ be a two dimensional torus acting on $\mathbb{P}^1$ by

$$(t_0, t_1).[z_0 : z_1] = [t_0 z_0 : t_1 z_1] \,.$$

(a) Compute the T-equivariant cohomology $H_T^*(\mathbb{P}^1)$.
(b) Compute

$$\int_{\mathbb{P}^1} c_1(T\mathbb{P}^1)$$

using the Atiyah–Bott localization formula.

18. Let $a \in \mathbb{Z}$. Compute the double ramification cycle $\mathrm{DR}_1(-a, a)$ on $\overline{\mathcal{M}}_{1,2}$ by Pixton's formula.

19. (a) Compute λ_2 in $H^4(\overline{\mathcal{M}}_{2,1})$ using Pixton's formula.
(b) Use (a) to compute

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi_1^2 \lambda_2 \,.$$

20. Let $\mathcal{M}^{\mathsf{ct}}_{g,n}$ be the open substack of $\overline{\mathcal{M}}_{g,n}$ consists of

$$(C, p_1, \ldots, p_n) \text{ where the dual graph does not have loops.}$$

This space is called the *moduli space of curves of compact type*. Prove the restriction of λ_g to $\mathcal{M}^{\mathsf{ct}}_{g,n}$ vanishes,

$$\lambda_g|_{\mathcal{M}^{\mathsf{ct}}_{g,n}} = 0 \text{ in } H^{2g}(\mathcal{M}^{\mathsf{ct}}_{g,n})\,.$$

Extra. Show that there exists a divisor Θ_A in $H^2(\mathcal{M}^{\mathsf{ct}}_{g,n})$ such that the Pixton's formula $\mathsf{P}_{g,A}$ restricted to $\mathcal{M}^{\mathsf{ct}}_{g,n}$ can be written as follows:

$$\mathsf{P}_{g,A} = \frac{(\Theta_A)^g}{g!} \text{ in } H^{2g}(\mathcal{M}^{\mathsf{ct}}_{g,n})\,.$$