5.3. An application of the Martingale Convergence Theorem.

Let's begin by recalling the construction of an infinite product of probability measure spaces. For every $n \geq 1$, let (Y_n, μ_n) be a probability measure space. Let

$$\Omega = \prod_{n \geq 1} Y_n$$

~~xxxx~~, $\Omega_i := \prod_{n \geq i+1} Y_n$, and $\mathcal{G}$ the σ-algebra of subsets of Ω generated by the subsets of the form $A \times \Omega_i$ where $A \subset Y_1 \times \cdots \times Y_i$ is $\mu_1 \times \cdots \times \mu_i$ measurable. Then there is a unique probability measure τ on Ω defined on $\mathcal{G}$ such that

$$\tau(A \times \Omega_i) = (\mu_1 \times \cdots \times \mu_i)(A)$$

for $A \times \Omega_i$ as above. (See Halmos, "measure theory" § 38 Thm 2). The following is a special case of the martingale convergence theorem, (see Loève, "Probability theory" 29.3):

<u>Thm 5.14.</u> $\forall n \geq 1$ let $f_n : \Upsilon_1 \times \cdots \times \Upsilon_n \to [0, \infty]$ be in $L^1(\mu_1 \times \cdots \times \mu_n)$. Assume that

$$f_n(y_1, \dots, y_n) = \int_{\Upsilon_{n+1}} f_{n+1}(y_1, \dots, y_n, y_{n+1})\, d\mu_{n+1}(y_{n+1})$$

$\forall n \geq 1$, and for almost all $(y_1, \dots, y_n) \in \Upsilon_1 \times \dots \times \Upsilon_n$. Then for τ-almost all sequences $(y_1, y_2, \dots) \in \Omega$, the limit

$$\lim_{n \to \infty} f_n(y_1, \dots, y_n)$$

exists.

Let us indicate the link with bounded harmonic functions.

Let G be l.c.c.c $\mu \in M^1(G)$ and

$$f : G \longrightarrow \mathbb{R}$$

bounded μ-harmonic. Let $\forall n \geq 1$,

$$f_n : G^n \longrightarrow \mathbb{R}$$

be $f_n(g_1, \dots, g_n) := f(g_1 \cdots g_n)$.

Then the μ-harmonicity of f implies

$$\int_G f_n(g_1, \dots, g_n, g_{n+1})\, d\mu(g_{n+1})$$

$$= \int_G f(g_1 \cdots g_n g_{n+1})\, d\mu(g_{n+1})$$

$$= f(g_1 \cdots g_n) = f_n(g_1, \dots, g_n).$$

By adding a suitable constant we can make f positive and the Martingale convergence theorem implies then that for τ-almost all sequence $(g_1, g_2, \dots) \in G^{\mathbb{N}}$

$$\lim_{n \to \infty} f(g_1 \cdots g_n) \text{ exists.}$$

It is then not astonishing that one construction of the Poisson boundary of (G, μ) proceeds by taking an appropriate quotient of $G^{\mathbb{N}}$ in the measure space category.

The application we have in view is the following. Let Γ be discrete countable, $\mu \in M^1(\Gamma)$, $\Gamma \times M \longrightarrow M$ a continuous action on a compact metrizable space M and $\nu \in M^1(M)$ a μ-stationary measure, which exists by Prop. 5.8. The following is due to Guivarc'h and Raugi ("Products of random matrices: convergence theorems", Contemp. Math. 1986).

Let $\Omega = \Gamma^{\mathbb{N}}$ with product measure $\tau = \mu^{\mathbb{N}}$ and

$$\lambda := \sum_{k \geq 1} \frac{1}{2^k} \mu^{*k}$$

Where $\mu^{*k} = \underbrace{\mu * \cdots * \mu}_{k}$.

<u>Prop. 5.15.</u> The sequence $\gamma_1 \cdots \gamma_n * \nu$ converges in $M^1(M)$ for τ-almost every $(\gamma_1, \gamma_2, \ldots) \in \Omega$ and we have for $\tau \times \lambda$ a.e. $((\gamma_1, \gamma_2, \ldots), \gamma) \in \Omega \times \Gamma$:

$$\lim_{n \to \infty} \gamma_1 \cdots \gamma_n \nu = \lim_{n \to \infty} \gamma_1 \cdots \gamma_n \gamma \nu .$$

<u>Proof</u>:

(1) The first assertion follows readily from the Martingale Convergence theorem as follows.

Let $f \in C(M)$; define

$$\hat{f}(g) = (g_* \nu)(f) = \int_M f(g\xi)\, d\nu(\xi).$$

Since ν is μ-stationary we have that

$\hat{f}$ is bounded ~~&~~ μ-harmonic and hence $\lim_{n\to\infty} \hat{f}(\gamma_1 \cdots \gamma_n)$ exists for τ-a.e. $(\gamma_1, \ldots) \in \Omega$; that is,

$$\lim_{n\to\infty} (\gamma_1 \cdots \gamma_n * \nu)(f)$$

exists.

Since M is metrisable compact, $C(M)$ is a separable Banach space (with sup. norm) and taking $\mathcal{D} \subset C(M)$ countable dense we have that a sequence $(\alpha_n)_{n\geq 1}$ in $M'(M)$ weak $*$-converges ~~to~~

~~iff $\alpha_n(f) \to \alpha(f)$ $\forall f \in \mathcal{D}$.~~

iff $(\alpha_n(f))_{n\geq 1}$ converges $\forall f \in \mathcal{D}$.

Thus if $E_f = \{ (\gamma_1, \ldots) \in \Omega :$

$\lim \gamma_1 \cdots \gamma_n * \nu(f)$ does

not exist $\}$

then $\tau(E_f) = 0$ and since $\mathcal{D}$ is countable, $\tau\left(\bigcup_{f \in \mathcal{D}} E_f\right) = 0$.

Thus $\forall\, (\gamma_1, \dots) \notin \bigcup_{f \in \mathcal{D}} E_f$ and for every $f \in \mathcal{D}$, $\lim_{n\to\infty} (\gamma_1 \cdots \gamma_n {}_* \nu)(f)$ exists and hence $\lim_{n\to\infty} \gamma_1 \cdots \gamma_n {}_* \nu$ exists.

(2) We show that for every $r \geq 1$ and for every $f \in C(M)$,

$$\lim_{n\to\infty} \left(\gamma_1 \cdots \gamma_n \gamma_* \nu(f) - \gamma_1 \cdots \gamma_n {}_* \nu(f) \right) = 0$$

for $\tau \times \mu^{*r}$ almost every $((\gamma_1, \dots), \gamma) \in \Omega \times \Gamma$.

It will be convenient to denote for $\omega = (\gamma_1, \dots) \in \Omega$, $\omega_n = \gamma_1 \cdots \gamma_n$. Setting as above $\hat{f}(\eta) = (\eta_* \nu)(f)$ we compute:

$$\int_\Omega \int_\Gamma (\gamma_1 \cdots \gamma_n \gamma \nu(f) - \gamma_1 \cdots \gamma_n \nu(f))^2 \, d\mu^{*r}(\gamma) \, d\tau(\omega)$$

$$= \int_\Omega \int_\Gamma (\hat{f}(\omega_n \gamma) - \hat{f}(\omega_n))^2 \, d\mu^{*r}(\gamma) \, d\tau(\omega).$$

$$= \int_\Omega \int_\Gamma \hat{f}(\omega_n \gamma)^2 \, d\mu^{*r}(\gamma) \, d\tau(\omega)$$

$$- 2 \int_\Omega \int_\Gamma \hat{f}(\omega_n \gamma) \hat{f}(\omega_n) \, d\mu^{*r}(\gamma) \, d\tau(\omega)$$

$$+ \int_\Omega \int_\Gamma \hat{f}(\omega_n)^2 \, d\mu^{*r}(\gamma) \, d\tau(\omega).$$

Taking into account that

$$\int_\Gamma \hat{f}(\eta \gamma) \, d\mu^{*}(\gamma) = \hat{f}(\eta)$$

We obtain :

$$= \int_\Omega \int_\Gamma \hat{f}(\omega_n \gamma)^2 \, d\mu^{*r}(\gamma) \, d\tau(\omega) - \int_\Omega \int_\Gamma \hat{f}(\omega_n) \, d\tau(\omega)$$

$$= \mu^{*n+r}(\hat{f}^2) - \mu^{*n}(\hat{f}^2).$$

For $p \geq r+1$ we compute:

$$I_{p,r} := \int_\Gamma d\mu^{*r}(\gamma) \int_\Omega \sum_{n=1}^{p} \left(\hat{f}(\omega_n \gamma) - \hat{f}(\omega_n)\right)^2 d\tau(\omega)$$

$$= \mu^{r+1}(\hat{f}^2) - \mu(\hat{f}^2)$$

$$+ \mu^{r+2}(\hat{f}^2) - \mu^{*2}(\hat{f}^2)$$

$$\cdots$$

$$+ \mu^{r+p}(\hat{f}^2) - \mu^{p}(\hat{f}^2)$$

$$= \left(\mu^{p+1}(\hat{f}^2) + \cdots + \mu^{p+r}(\hat{f}^2)\right)$$

$$- \left(\mu(\hat{f}^2) + \cdots + \mu^{r}(\hat{f}^2)\right)$$

Bref: $\mu^{*k}(\hat{f}^2) \leq \|\hat{f}^2\|_\infty \leq \|f\|_\infty^2$.

Hence $I_{p,r} \leq 2r \|f\|_\infty^2$.

Dividing by 2^r and summing up we get:

$$\int_\Gamma d\lambda(\gamma) \int_\Omega \sum_{n=1}^{p} \left(\hat{f}(\omega_n \gamma) - \hat{f}(\omega_n)\right)^2 d\tau(\omega)$$

$$\leq \underbrace{\left(\sum_{r=1}^{\infty} \frac{1}{2^r} \cdot 2 \cdot r\right)}_{4} \|f\|_\infty^2$$

$$= 4 \|f\|_\infty^2 .$$

Applying the monotone convergence theorem twice we get

$$\int_\Gamma d\lambda(\gamma) \int_\Omega \sum_{n=1}^{\infty} \left(\hat{f}(\omega_n \gamma) - \hat{f}(\omega_n)\right)^2 d\tau(\omega)$$

$$\leq 4 \|f\|_\infty^2 .$$

By a theorem of Lebesgue this implies that

$$\sum_{n=1}^{\infty} \left(\hat{f}(\omega_n \gamma) - \hat{f}(\omega_n)\right)^2 < +\infty$$

for $\tau \times \lambda$ - a.e. (ω, γ) which implies (2).

$\boxed{\S}$

A Reduction.

5.5. ~~[illegible]~~

Our objective in this ~~section~~ and the next section is to prove Thm 5.11; ~~[illegible]~~ the main step towards this goal will be:

Thm 5.16.

Assume $\rho : \Gamma \longrightarrow GL(V)$ is strongly irreducible, proximal and $\mu \in M^1(\Gamma)$ admissible. Then for every μ-stationary measure $\nu \in M^1(\mathbb{P}V)$ we have

$$\lim_{n \to \infty} g_1 \cdots g_n \nu \in \delta_{\mathbb{P}(V)}$$

for τ-almost every $(g_1, \dots) \in \Omega = \Gamma^{\mathbb{N}}$.

Remark that by Thm 5.15 we already know that $\lim_{n \to \infty} g_1 \cdots g_n \nu \in M^1(\mathbb{P}V)$ exists for τ-a.e. $(g_1, g_2, \dots)$

so the point of the Thm is that these limits are Dirac masses: that's where the hypothesis on ρ enters.

First, as a motivation, let's show how Thm 5.16 implies Thm 5.11; it also provides an a posteriori justification for the approach via Martingale convergence theorem.

We will have to study probability measures on $M^1(\mathbb{P}V)$ that is elements $\sigma \in M^1(M^1(\mathbb{P}V))$. Such a probability measure has a center of mass

$$c(\sigma) := \int_{M^1(\mathbb{P}V)} y \, d\sigma(y).$$

We let d be a distance inducing the weak-$*$-topology on $M^1(\mathbb{P}V)$. For instance if $\{f_n : n \geq 1\}$ is a dense countable subset of the unit ball of $C(\mathbb{P}V)$ then

$$d(\mu,\nu)=\sum_{k=1}^{\infty}\frac{|\mu(f_k)-\nu(f_k)|}{2^k}$$

is such a distance.

<u>Lemma 5.17.</u> Let $\sigma\in M^1(M^1(\mathbb{P}V))$ and

$$c(\sigma)=\int_{M^1(\mathbb{P}V)} y\,d\sigma(y).$$

Assume that for some sequence $(\gamma_n)_{n\geq 1}$ in Γ $\lim_{n\to\infty}\gamma_n c(\sigma)=\delta_x$ for some $x\in\mathbb{P}V$.

Then: $\forall\varepsilon>0$,

$$\sigma\{y\in M^1(\mathbb{P}V):d(\gamma_n y,\delta_x)>\varepsilon\}\xrightarrow[n\to\infty]{}0.$$

<u>Proof</u>: We have:

$$\gamma_n c(\sigma)=\int_{M^1(\mathbb{P}V)}\gamma_n y\,d\sigma(y)$$

Now the geometric explanation of the lemma is that as n gets large $\gamma_n c(\sigma)$ is near δ_x which is an extremal point of the convex set $M^1(\mathbb{P}V)$; as a result σ must give small measure to the set $\{y \in M^1(\mathbb{P}V) : d(\gamma_n y, \delta_x) > \varepsilon\}$.

Now formally the proof goes as follows.

Let $f \in C(\mathbb{P}V)$ with $f \geq 0$ and $f(x) = 0$.

Then:

$$(\gamma_n c(\sigma))(f) = \int_{M^1(\mathbb{P}V)} (\gamma_n y)(f)\, d\sigma(y)$$

$$\geq \varepsilon \cdot \sigma\{y \in M^1(\mathbb{P}V) : (g_n y)(f) > \varepsilon\}.$$

Since $(\gamma_n c(\sigma))(f) \xrightarrow[n\to\infty]{} \delta_x(f) = f(x) = 0$ this implies

$$\sigma\{y \in M^1(\mathbb{P}V) : (\gamma_n y)(f) > \varepsilon\} \xrightarrow[n\to\infty]{} 0.$$

Using the above distance we have

$$d(\gamma_n y, \delta_x) = \sum_{k=1}^{\infty} \frac{|\gamma_n y(f_k) - f_k(x)|}{2^k}$$

$$\leq \sum_{k=1}^{\infty} \frac{\gamma_n y(|f_k - f_k(x)|)}{2^k}$$

$$= \gamma_n y \Bigg(\underbrace{\sum_{k=1}^{\infty} \frac{|f_k - f_k(x)|}{2^k}}_{=: f} \Bigg)$$

Thus

$$\sigma\{y : d(\gamma_n y, \delta_x) > \varepsilon\} \leq \sigma\{y : \gamma_n y(f) > \varepsilon\} \xrightarrow{n \to \infty} 0 .$$

$\boxed{\text{≷}}$

To connect this with Thm 5.11, recall that the in Thm 5.11 we have

$\rho : \Gamma \to GL(V)$ strongly irreducible, proximal

$\mu \in M^1(\Gamma)$ admissible, $\Gamma \times B \to B$ a continuous action on a locally compact metrisable space B, $\nu \in M^1(B)$ a μ-stationnary measure and

$$\psi : B \to M^1(\mathbb{P}V)$$

an equivariant measurable map. Then

$$\sigma := \psi_*(\nu) \in M^1(M^1(\mathbb{P}V))$$

and σ is μ-stationnary. Theorem 5.11 will then follow from

<u>Lemma 5.18</u> ~~Under the hypothesis of~~ Assuming Thm 5.16 let $\sigma \in M^1(M^1(\mathbb{P}V))$ be μ-stationnary. Then

$$\text{support}(\sigma) \subset \delta_{\mathbb{P}(V)} = \{\delta_x : x \in \mathbb{P}(V)\}.$$

<u>Proof:</u> Let $\nu := c(\sigma) = \int_{M^1(A(V))} y \, d\sigma(y)$.

Then ν is μ-stationnary and hence by Thm 5.16,

$$\lim_{n \to \infty} g_1 \cdots g_n \nu \in \delta_{A(V)}$$

for τ-a.e. $(g_1, \dots) \in \Omega = \Gamma^{\mathbb{N}}$.

It will be convenient to set for $w \in \Omega$, $w = (g_1, \dots, \dots)$, $w_n = g_1 \cdots g_n$.

So by lemma 5.17 we have:

$$\lim_{n \to \infty} \sigma \{ y \in M^1(A(V)) : d(w_n y, \delta_{A(V)}) > \varepsilon \}$$

$$= 0$$

for τ-a.e. $w \in \Omega$. Consider now:

$$E_n = \{ (w, y) \in \Omega \times M^1(A(V)) :$$

$$d(w_n y, \delta_{A(V)}) > \varepsilon \}.$$

By dominated convergence and Fubini we get:

$$\lim_{n\to\infty} (\tau \times \sigma)(E_n) = \lim_{n\to\infty} \int_{\Omega} d\tau(\omega) \int_{M^1(P(V))} \chi_{E_n}((\omega, y))\, d\sigma(y)$$

$$= \int_{\Omega} d\tau(\omega) \underbrace{\lim_{n\to\infty} \int_{M^1(P(V))} \chi_{E_n}((\omega, y))\, d\sigma(y)}_{\substack{\shortparallel \\ 0}}$$

$$= 0.$$

Now consider the map

$$\alpha_n : \Omega \times M^1(P(V)) \longrightarrow M^1(P(V))$$
$$(\omega, y) \longmapsto \omega_n \cdot y$$

Since σ is μ-stationnary, we have

$$\mu * \dots * \mu * \sigma = \sigma$$

which translates to:

$$(\alpha_n)_* (\tau \times \sigma) = \sigma.$$

Now

$$E_n = \alpha_n^{-1}\underbrace{\{y \in M^1(B(V)) : d(y, \delta_{B(V)}) > \varepsilon\}}_{S_\varepsilon}$$

Thus

$$0 = \lim_{n\to\infty} (\tau \times \sigma)(E_n)$$

$$= \lim_{n\to\infty} (\tau \times \sigma)(\alpha_n^{-1}(S_\varepsilon))$$

$$= \lim_{n\to\infty} (\alpha_n)_*(\tau \times \sigma)(S_\varepsilon)$$

$$= \sigma(S_\varepsilon)$$

That is $\sigma\{y \in M^1(B(V)) : d(y, \delta_{B(V)}) > \varepsilon\} = 0$

$\forall \varepsilon > 0$ which shows the lemma. $\boxed{\Sigma}$