5.6. Dynamics on $\mathbb{P}(V)$ and the proof of Thm 5.16.

Recall that V is a finite dimensional k-vector space where k is a local field of characteristic zero. Recall that as such k is locally compact, all field operations are continuous and k is endowed with a canonical norm

$$|\,| : k \longrightarrow \mathbb{R}_{\geq 0}$$

satisfying $|x| \geq 0$ with equality iff $x = 0$, $|x \cdot y| = |x|\,|y|$ and $|x+y| \leq |x| + |y|$. Actually the classification of such k's tells us that $k = \mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$. In the latter case, the triangle inequality takes the stronger form $|x+y| \leq \max(|x|, |y|)$.

If $e_1, \dots, e_d$ is a basis of V we can define the norm of $x \in V$, by

$$\|x\| = \max_{1 \le i \le d} |x_i|$$

Where $x = \sum_{i=1}^{d} x_i e_i$.

All such norms are equivalent; they make V locally compact and the quotient space

$$\mathbb{P}(V) = k^{\times} \backslash V$$

is compact metrisable. We then endow $\mathrm{End}(V)$ with a norm:

$$\|u\| = \sup_{\|x\| \le 1} \|u(x)\|.$$

This gives a topology on $\mathrm{End}(V)$ which coincides with the topology of pointwise convergence. All this is well known for $k = \mathbb{R}$ and the reader can just stick to this case.

Every $g \in GL(V)$ induces a map

$$\bar{g} : \mathbb{P}(V) \to \mathbb{P}(V) \ ;$$

we can extend this to $\mathrm{End}(V) \setminus (0)$ in the following way: if $t \in \mathrm{End}(V)$, $t \neq 0$, then t induces a well defined map

$$\bar{t} : \mathbb{P}(V) \setminus \mathbb{P}(\mathrm{Ker}\, t) \longrightarrow \mathbb{P}(V)$$

called quasiprojective transformation. The following simple lemma will be a guiding principle for everything we do.

<u>Lemma 5.13</u>. Let $(g_n)_{n \geq 1}$ be a sequence in $GL(V)$. Then there exists a subsequence $(g_{n_k})_{k \geq 1}$, a sequence $\lambda_k \in k$, $\lambda_k \neq 0$ and $u \in \mathrm{End}(V)$, $u \neq 0$ such that in $\mathrm{End}(V)$, $\lim_{k \to \infty} \frac{g_{n_k}}{\lambda_k} = u$.

In particular the sequence $(\overline{g}_{n_k})_{k\geq 1}$ converges pointwise on $\mathbb{P}(V) \setminus \mathbb{P}(\operatorname{Ker} u)$ to the quasi projective map

$$\overline{u} : \mathbb{P}(V) \setminus \mathbb{P}(\operatorname{Ker} u) \longrightarrow \mathbb{P}(V).$$

<u>Proof</u>: Chose $\mu_n \in k^\times$ such that $\left\| \frac{g_n}{\mu_n} \right\| = 1 \ \forall n \geq 1$. But now

$$\{ t \in \operatorname{End}(V) : \|t\| = 1 \}$$

is compact metrisable; thus there is a subsequence $\left(\frac{g_{n_k}}{\mu_{n_k}} \right)_{k\geq 1}$ that converges to an endomorphism $u \in \operatorname{End}(V)$ with $\|u\| = 1$; in particular $u \neq 0$. The rest is immediate. ∎

This motivates the following definition:

<u>Def. 5.20.</u> The sequence $(g_n)_{n\geq 1}$ in $GL(V)$ is contracting if there is $\lambda_n \neq 0$ $\forall n \geq 1$ and $u \in End(V)$ of <u>rank 1</u>, such that $\lim_{n\to\infty} \frac{g_n}{\lambda_n} = u$.

Recall that a subgroup $\Lambda < GL(V)$ is strongly irreducible if every finite index subgroup of Λ acts irreducibly on V, and proximal if Λ acts proximally on $\mathbb{P}(V)$. Our first objective is to prove:

<u>Prop. 5.21.</u> Let $\Lambda < GL(V)$ be strongly irreducible. TFAE:

(1) Λ is proximal.

(2) Λ contains a contracting sequence.

For this we will need two lemmas.

<u>Lemma 5.22.</u> Assume Λ acts proximally on $\mathbb{P}(V)$. Then for every finite subset $\{x_1, .., x_\ell\} \subset \mathbb{P}(V)$ there exists $z \in \mathbb{P}(V)$ and a sequence $(g_n)_{n\geq 1}$ in Λ with:

$$\lim_{n\to\infty} \bar{g}_n(x_i) = z \qquad 1 \leq i \leq \ell.$$

<u>Proof:</u>

For $\ell = 2$ this follows from proximality. Indeed if d is a distance on $\mathbb{P}(V)$, we know that there is a sequence $(\lambda_n)_{n\geq 1}$ in Λ with $\lim_{n\to\infty} d(\bar{\lambda}_n(x_1), \bar{\lambda}_n(x_2)) = 0$.

Since $\mathbb{P}(V)$ is compact we can extract a convergent subsequence:

$$\bar{\lambda}_{n_k}(x_1) \underset{k\to\infty}{\longrightarrow} z.$$

But then $\bar{\lambda}_{n_k}(x_2) \longrightarrow z$.

The general case is an easy induction on l and left to the reader. $\square$

The next lemma uses some elementary facts from the theory of linear algebraic groups: it would be interesting to have an elementary proof.

<u>Lemma 5.23.</u> Let $\Lambda < GL(V)$ be strongly irreducible, $L < V$ a vector subspace with $1 \leq \dim L \leq \dim V - 1$ and $a, b \in \mathbb{P}(V)$. Then there exists $\lambda \in \Lambda$ with $\lambda(a) \notin \mathbb{P}(L)$ and $\lambda(b) \notin \mathbb{P}(L)$.

<u>Proof:</u>

Let $\bar{k}$ be an algebraic closure of k and for any vector subspace $W \subset V$ let $W_{\bar{k}}$ denote the corresponding $\bar{k}$-points.

Let $\mathbb{H} < GL(V_{\bar{k}})$ be the Zariski closure of Λ and $\mathbb{H}^\circ$ its connected component of the identity. As $\mathbb{H}^\circ$ is of finite index in $\mathbb{H}$, so is $\Lambda' := \mathbb{H}^\circ \cap \Lambda$ in Λ and hence Λ' acts irreducibly on V.

Consider $\mathbb{H}_a = \{ h \in \mathbb{H}^\circ : h(a) \in \mathbb{P}(L_{\bar{k}}) \}$

$$\mathbb{H}_b = \{ h \in \mathbb{H}^\circ : h(b) \in \mathbb{P}(L_{\bar{k}}) \}.$$

Observe that $\mathbb{H}_a \neq \mathbb{H}^\circ$, similarly $\mathbb{H}_b \neq \mathbb{H}^\circ$. Otherwise, if $\mathbb{H}_a = \mathbb{H}^\circ$ we would have

$$\forall \lambda \in \Lambda' : \quad \lambda(a) \in \mathbb{P}(L) ,$$

but then the vector subspace of L generated by the set of lines

$$\{ \lambda(a) : \lambda \in \Lambda' \}$$

would be a proper, nonzero Λ'-invariant subspace of V contradicting irreducibility.

Thus $\mathbb{H}_a$ and $\mathbb{H}_b$ are Zariski closed proper subsets of the connected algebraic group $\mathbb{H}^0$; since connected algebraic groups are irreducible,

$$\mathbb{H}_a \cup \mathbb{H}_b \neq \mathbb{H}^0.$$

But then $\mathbb{H}^0 \setminus (\mathbb{H}_a \cup \mathbb{H}_b)$ is a non-empty Zariski open subset of $\mathbb{H}^0$ hence of $\mathbb{H}$ and since Λ is Zariski dense in $\mathbb{H}$ we conclude

$$\Lambda \cap (\mathbb{H}^0 \setminus (\mathbb{H}_a \cup \mathbb{H}_b)) \neq \phi$$

which concludes the proof. ☒

<u>Proof of Prop. 5.21.</u>

(1) ⟹ (2) Observe first that $\forall N \geq d = \dim V$, there exists $x_1, \ldots, x_N$ in V such that any d of them span V.

By recurrence: for $N = d$ it is true.
Assume $N \geq d$ and $\{x_1, \ldots, x_N\}$ has this property. For every subset

$$F \subset \{x_1, \ldots, x_N\}$$

with $|F| = d-1$, let L_F be the linear span of F. Then $\dim L_F = d-1$ and

$$\bigcup_{|F|=d-1} L_F \neq V.$$

Now pick $x_{N+1} \in V \setminus \left(\bigcup_{|F|=d-1} L_F\right)$.

Then $\{x_1, \ldots, x_{N+1}\}$ satisfies the claimed property.

Now pick such a subset $\{x_1, \ldots, x_N\}$ with $N = 2d-1$. By lemma 5.22 there exists $(g_n)_{n\geq 1}$ in Λ and $z \in P(V)$ with

$$\lim_{n\to\infty} \bar{g}_n(x_i) = z \qquad 1 \leq i \leq 2d-1.$$

By lemma 5.17, let $\frac{g_{n_k}}{\lambda_{n_k}} \longrightarrow u$ be a convergent subsequence. Since $u \neq 0$, Ker u contains at most $d-1$ vectors in $\{x_1, \dots, x_{2d-1}\}$. Thus there is a basis $x_{i_1}, \dots, x_{i_d}$ of V with $x_{i_\ell} \notin$ Ker u, $1 \leq \ell \leq d$. Then:

$$z = \lim_{k \to \infty} \bar{g}_{n_k}(x_{i_\ell}) = \bar{u}(x_{i_\ell})$$

which implies that u has rank 1.

(2) ⟹ (1). Let $(g_n)_{n \geq 1}$ be a contracting sequence, that is $\frac{g_n}{\lambda_n} \longrightarrow u$, $u \neq 0$ and u has rank 1. Let $L = \operatorname{Ker} u$, $z = \operatorname{Im} u \in \mathbb{P}(V)$. Given $a, b \in \mathbb{P}(V)$, use lemma 5.23 to obtain $h \in \Lambda$ with $\bar{h}(a) \notin \mathbb{P}(L)$, $\bar{h}(b) \notin \mathbb{P}(L)$. Then

$$\lim_{n\to\infty} \overline{g_n h}(a) = z = \lim_{n\to\infty} \overline{g_n h}(b)$$

in particular, $\lim_{n\to\infty} d\left(\overline{g_n h}(a), \overline{g_n h}(b)\right) = 0$

and we are done. ∎

We need two more lemmas on measures on $\mathbb{P}(V)$. We will need to consider

$$M^1(\mathbb{P}(V))_0 = \{ \nu \in M^1(\mathbb{P}(V)) : \nu(\mathbb{P}(L)) = 0 \ \forall \text{ proper linear subspace } L \subsetneq V \}.$$

Observe that if $u \in \mathrm{End}(V)$, $u \neq 0$ then for $\nu \in M^1(\mathbb{P}(V))_0$ the direct image of ν under $\overline{u}$ is well defined and still a probability measure. In formula: for $f \in C(\mathbb{P}(V))$; $L = \operatorname{Ker} u$,

$$\overline{u}_*(\nu)(f) = \int_{\mathbb{P}(V)\setminus\mathbb{P}(L)} f(\overline{u}(\xi))\, d\nu(\xi)$$

$$= \int_{\mathbb{P}(V)} f(\overline{u}(\xi))\, d\nu(\xi)$$

since $\nu(\mathbb{P}(L)) = 0$.

Moreover $\forall g \in GL(V)$:

$$(\overline{ug})_*\nu = \overline{u}_*\, g_*(\nu).$$

<u>Lemma 5.24.</u> Assume $(g_n)_{n\geq 1}$ is a sequence in $GL(V)$ with $\frac{g_n}{\lambda_n} \longrightarrow u$, $u \in \operatorname{End}(V)$, $u \neq 0$. Let $\nu \in M^1(\mathbb{P}(V))_0$.

Then $\lim_{n\to\infty} \overline{g}_{n*}(\nu) = \overline{u}_*(\nu)$.

<u>Proof:</u> Let $L = \operatorname{Ker} u$, $f \in C(\mathbb{P}(V))$.

Then $\overline{g}_{n*}(\nu)(f) = \int_{\mathbb{P}(V)} f(\overline{g}_n(\xi))\, d\nu(\xi)$

$$= \int_{\mathbb{P}(V) \setminus \mathbb{P}(L)} f(\bar{g}_n(\xi))\, d\nu(\xi)$$

But $\lim_{n \to \infty} f(\bar{g}_n(\xi)) = f(\bar{u}(\xi))$; $\forall \xi \notin \mathbb{P}(L)$

hence the result follows by dominated convergence thm. ∎

The following lemma ~~is reminiscent of~~ the maximum principle for harmonic functions.

<u>Lemma 5.25.</u> Let $\rho : \Gamma \to GL(V)$ be strongly irreducible and $\mu \in M^1(\Gamma)$ admissible. If $\nu \in M^1(\mathbb{P}(V))$ is μ-stationary then $\nu(\mathbb{P}(L)) = 0$ $\forall$ linear subspace $L < V$ with $1 \leq \dim L \leq \dim V - 1$. That is, $\nu \in M^1(\mathbb{P}(V))_0$.

Proof:

Let $\lambda = \sum_{k \geq 1} \frac{1}{2^k} \mu^{*k}$. Then ν is λ-stationnary and $supp(\lambda) = \Gamma$.

Let $r = \min\{\dim W : 0 \neq W \subset V$ is a linear subspace such that $\nu(\mathbb{P}(W)) > 0\}$.

Given any $\delta > 0$, the set

$$\{L : \dim L = r,\ \nu(\mathbb{P}(L)) \geq \delta\}$$

is finite since by minimality of r we have $\nu(\mathbb{P}(L_1) \cap \mathbb{P}(L_2)) = 0 \quad \forall L_1 \neq L_2$ $\dim L_1 = \dim L_2 = r$. Thus there is W with $\dim W = r$ and

$$\nu(\mathbb{P}(W)) \geq \nu(\mathbb{P}(L)) \quad \forall L, \dim L = r.$$

Now consider $f(g) := \delta(g)_* \nu(\mathbb{P}(W))$.

Since $\lambda * \nu = \nu$, we have in particular that:

$$f(e) = \sum_{g \in \Gamma} \lambda(g) f(g).$$

Since λ is a probability measure and

$$f(g) \leq f(e) \quad \forall g \in \Gamma$$

AND since $\lambda(g) > 0 \quad \forall g \in \Gamma$

We obtain:

$$f(e) = f(g) \quad \forall g \in \Gamma.$$

Hence $\nu(\rho(g)\mathbb{P}(W)) = \nu(\mathbb{P}(W)) \quad \forall g \in \Gamma.$

This implies that

$$\{\rho(g)\mathbb{P}(W) : g \in \Gamma\}$$

is finite and hence there is $\Gamma' < \Gamma$ of finite index such that W is Γ'-invariant.

Hence $W = V$. $\blacksquare$

Proof of Thm 5.16.

By prop. 5.15 there is a subset $E \subset \Omega = \Gamma^{\mathbb{N}}$ of τ-measure 1 such that for every $\omega = (g_1, \dots) \in E$

$$\theta(\omega) := \lim_{n \to \infty} \rho(g_1) \cdots \rho(g_n)_* \nu \in M^1(P(V))$$

exists and

$$\lim_{n \to \infty} \rho(g_1) \cdots \rho(g_n) \rho(\gamma)_* \nu = \theta(\omega) \quad \forall \gamma \in \Gamma.$$

Now fix an $\omega \in E$. By lemma 5.19 take a subsequence

$$\frac{\rho(g_1) \cdots \rho(g_{n_k})}{\lambda_k} \longrightarrow \tau(\omega) \neq 0$$

in End V.

By lemma 5.24 and 5.25:

$$\lim \rho(g_1) \cdots \rho(g_{n_k})_* \nu = \overline{\tau(\omega)}_* \nu$$

$$\lim \rho(g_1) \cdots \rho(g_{n_k}) \rho(\gamma)_* \nu = \overline{\tau(\omega)}_* \rho(\gamma)_* \nu.$$

which implies:

$$\overline{\tau(\omega)_*\nu} = \overline{\tau(\omega)_*\rho(\gamma)_*\nu} \quad \forall \gamma \in \Gamma.$$

Let $W = \operatorname{Ker} \tau(\omega)$. We claim that there is a sequence $(\gamma_n)_{n\geq 1}$ in Γ and a point $z \notin \mathbb{P}(W)$ such that $(\rho(\gamma_n))_{n\geq 1}$ is contracting and

$$\lim_{n\to\infty} \rho(\gamma_n)_*\nu = \delta_z .$$

By Prop. 5.21 we know that there is a ~~contracting~~ sequence $(\gamma_n)_{n\geq 1}$ with $(\rho(\gamma_n))_{n\geq 1}$ contracting. By lemma 5.24 we have $\lim_{n\to\infty} \rho(\gamma_n)_*\nu = \delta_z$ for some $z \in \mathbb{P}(V)$. But if now for every contracting sequence as above we have $z \in \mathbb{P}(W)$ then $\rho(\gamma\gamma_n)_{n\geq 1}$ is contracting as well and hence $\lim_{n\to\infty} \rho(\gamma\gamma_n)_*\nu = \rho(\gamma)\delta_z = \delta_{\rho(\gamma)z}$

thus $\rho(\gamma)z \in \mathbb{P}(W)$ $\forall \gamma \in \Gamma$ which contradicts irreducibility.

Thus let $(\gamma_n)_{n\geq 1}$ be a sequence such that $\lim_{n\to\infty} \frac{\rho(\gamma_n)}{\lambda_n} = u$, u has rank 1, $z := \operatorname{Im} u \notin \mathbb{P}(W) = \mathbb{P}(\operatorname{Ker}\tau(\omega))$

Then,

$$\tau(\omega)\rho(\gamma_n)_*\nu(f) = \int_{\mathbb{P}(V)} f\left(\overline{\tau(\omega)\rho(\gamma_n)\xi}\right) d\nu(\xi)$$

$$= \int_{\mathbb{P}(V)\setminus\mathbb{P}(\operatorname{Ker} u)} f\left(\overline{\tau(\omega)\rho(\gamma_n)\xi}\right) d\nu(\xi)$$

For every $\xi \notin \mathbb{P}(\operatorname{Ker} u)$ $\lim_{n\to\infty}\rho(\gamma_n)\xi = z$.

Since $z \notin \mathbb{P}(\operatorname{Ker}\tau(\omega))$ ~~we have~~ and $\mathbb{P}(V)\setminus\mathbb{P}(\operatorname{Ker}\tau(\omega))$ is open, we have that for every ~~these~~ $\xi \notin \mathbb{P}(\operatorname{Ker} u)$

there is $n(\xi)$ such that $\forall n \geq n(\xi)$

$$\rho(\gamma_n)\xi \in \mathbb{P}(V) \setminus \mathbb{P}(\operatorname{Ker} \tau(\omega)).$$

Now $\overline{\tau(\omega)} : \mathbb{P}(V) \setminus \mathbb{P}(\operatorname{Ker}\tau(\omega)) \longrightarrow \mathbb{P}(V)$

is continuous and hence

$$\lim_{n\to\infty} \overline{\tau}(\omega)\rho(\gamma_n)\xi = \overline{\tau(\omega)}\cdot\xi$$

$\forall \xi \notin \mathbb{P}(\operatorname{Ker} u)$.

By dominated convergence we get

$$\left(\overline{\tau(\omega)}_* \rho(\gamma_n^{-1})_* \nu\right)(f) \longrightarrow f(\overline{\tau(\omega)}\xi)$$

and hence $\Phi(\omega) = \delta_{\overline{\tau(\omega)}\xi}$.

∎