Chapter 6 The superrigidity theorem.

Let's first introduce some objects in order to state the main theorem. The adjoint representation

$$\mathrm{Ad} : GL(V) \longrightarrow GL(\mathrm{End}\, V)$$
$$g \longmapsto \mathrm{Ad}(g)$$

where $\mathrm{Ad}(g)(u) = gug^{-1}$, $u \in \mathrm{End}\, V$, will play an important role.

<u>Thm 6.1.</u> Let $\Gamma < SL(n,\mathbb{R})$ be a lattice and $\rho : \Gamma \longrightarrow GL(V)$ a strongly irreducible proximal representation. Assume $n \geq 3$, and $\dim V \geq 2$. Then $k = \mathbb{R}$ or $\mathbb{C}$, and there exist $T : SL(n,\mathbb{R}) \longrightarrow GL(I)$ an almost faithfull continuous repre-

sentation, $\mathcal{J} \subset \mathrm{End}(V)$ an $\mathrm{Ad}\circ\rho(\Gamma)$ invariant linear subspace and a linear isomorphism $\varepsilon : \mathcal{I} \longrightarrow \mathcal{J}$ intertwining $T|_\Gamma$ and $\mathrm{Ad}\circ\rho$, that is:

$$\varepsilon \circ T(\gamma) = \mathrm{Ad}(\rho(\gamma)) \circ \varepsilon \quad \forall \gamma \in \Gamma.$$

This theorem together with some expertise in algebraic groups implies the more familiar form of superrigidity

<u>Thm 6.2.</u> Let $\Gamma < SL(n,\mathbb{R})$ be a lattice $n \geq 3$ and $\mathbb{H}$ a connected k-almost simple algebraic group where k is a local field of characteristic zero. Let

$$\rho : \Gamma \longrightarrow \mathbb{H}_k$$

be a homomorphism with Zariski

dense image. Assume $\rho(\Gamma) \subset \mathbb{H}_k$ is not relatively compact. Then $k = \mathbb{R}, \mathbb{C}$ and ρ extends to an algebraic homomorphism $SL(n, \mathbb{R}) \longrightarrow \mathbb{H}_k / Z(\mathbb{H}_k)$.

In fact the connection between the two theorems is given by

<u>Lemma 6.3.</u> (see Margulis, book VI, 4.5)
With $\mathbb{H}$ and k as above, let $F < \mathbb{H}_k$ be a Zariski dense not relatively compact subgroup. Then there is a representation $\rho : \mathbb{H}_k \longrightarrow GL(W)$ such that $\rho(F)$ is strongly irreducible proximal.

A basic theorem about lattices going back to the 60's is the following

<u>Thm 6.4.</u> (Borel density) A lattice $\Gamma < SL(n,\mathbb{R})$ is Zarski dense.

In other words any polynomial $P \in \mathbb{R}[x_{ij}]$ in matrix entries x_{ij} that vanishes on Γ vanishes on $SL(n,\mathbb{R})$. There is now a rather simple proof available that uses concepts from ergodic theory we have treated in Chapter 4. We will come back to this later.

First I want to give an overview on the strategy of the proof of Thm 6.1. In the sequel $G = SL(n,\mathbb{R})$, $P = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$,

and $A = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \prod \lambda_i = 1, \lambda_i > 0 \right\}$.

<u>Step 1.</u> : Using that $\rho : \Gamma \to GL(V)$ and its contragradient $\rho^* : \Gamma \longrightarrow GL(V^*)$ are strongly irreducible (see Lemma 6.5 below), together with chapter 4, we show the existence of

$$E : G/A \longrightarrow End(V)$$

such that (1) E is $Ad\circ\rho - \Gamma$- equivariant measurable.

(2) for a.e. $x \in G/A$, $E(x) \in End(V)$ is a rank 1 endomorphism.

<u>Step 2.</u> : let $F_\Gamma(G, End(V))$ be the vector space of (classes) of measurable $Ad\rho - \Gamma$- equivariant maps with the topology of convergence in measure (see ....).

Define $T : G \longrightarrow GL(F_\Gamma(G, \mathrm{End} V))$ by $(T(g) f)(h) = f(hg)$. Notice that we can view E as a vector in $F_\Gamma(G, \mathrm{End} V)$ satisfying $T(a) E = E$ $\forall a \in A$.

Here is the main technical part:

<u>Main Lemma</u> (see) Let $S < A$ be closed non-compact and

$$I \subset F_\Gamma(G, \mathrm{End} V)$$

a finite dimensional $T(S)$-invariant subspace. Then

$$\mathrm{Lin}\{ T(c) f : f \in I, c \in Z_G(S) \}$$

is finite dimensional.

This will use repeatedly the ergodicity properties established in Chap 4.

Step 3 : now is the first ~~[illegible]~~ time we use the hypothesis $n \geq 3$. We show that there are closed subgroups $S_1, \dots, S_\ell$ in A that are non-compact (in fact isomorphic to $\mathbb{R}^\times_{>0}$) such that:

$$SL(n,\mathbb{R}) = Z_G(S_1) \cdot \dots \cdot Z_G(S_\ell).$$

This in itself is just manipulations with matrices. But it is equivalent to saying that the spherical building associated to $SL(n,\mathbb{R})$ is connected.

Step 4 : We deduce from Step 2 and 3 that when $n \geq 3$,

$$I := \mathrm{Lin}\{ T(g) E : g \in G \}$$

is a finite dimensional subvector space of $F_\Gamma(G, \mathrm{End}\, V)$. We conclude that the restriction $\begin{array}{ccc} G & \longrightarrow & GL(I) \\ g & \longmapsto & T(g)|_I \end{array}$

gives a measurable group homomorphism into the locally compact (!) group $GL(I)$.

Step 5: the map $E : G \longrightarrow \mathrm{End}(V)$ coincides almost everywhere with a unique Ad_S-Γ-equivariant continuous map.

This follows from Step 4 and a thm. of Mackey that says that a measurable hom. $m : G \longrightarrow H$ where G is l.c. s.c. and H is second countable coincides a.e. with a continuous homomorphism.

Step 6: By the above we can identify I with a space of strictly Ad_S-Γ-equivariant continuous maps $G \to \mathrm{End}(V)$. The map

$$\mathcal{E} : \mathcal{I} \to \mathrm{End}(V)$$
$$f \mapsto f(e)$$

is then shown to be injective, by the use of Borel density, and $\mathcal{J} := \mathcal{E}(\mathcal{I})$.

6.1. Step 1.

We outline the construction of the map $E : G/A \to \mathrm{End}(V)$. It will be obtained as the composition of the following maps:

$$G/A \longrightarrow G/P \times G/\bar{P} \xrightarrow[\varphi \times \varphi^*]{} \mathbb{P}(V) \times \mathbb{P}(V^*) \rightarrowtail \mathrm{End}(V)$$

The first map sends G/A to a G-orbit of full measure in $G/P \times G/P$, where G acts diagonally on $G/P \times G/P$. The last one is a partially defined map which stems from

from the fact that ~~given~~ to $v \in V$ and $\lambda \in V^*$ with $v \notin \operatorname{Ker}\lambda$, one can associate a rank 1 endomorphism with image $\mathbb{R}\cdot v$.

Let's concentrate on the second map.

Let thus $\rho : \Gamma \to GL(V)$ be strongly irreducible proximal. Recall that the contragredient representation $\rho^* : \Gamma \to GL(V^*)$ is defined by:

$$\rho^*(\gamma)(\lambda)(v) = \lambda(\rho(\gamma)^{-1}v).$$

<u>Lemma 6.5.</u> If $\rho : \Gamma \to GL(V)$ is strongly irreducible proximal, then so is $\rho^* : \Gamma \to GL(V^*)$.

<u>Proof:</u> This follows rather directly from Prop. 5.21. First, since ρ is strongly irreducible, then so is ρ^*, since if $W \subset V^*$ is invariant under $\underset{f.i.}{\Gamma' < \Gamma}$ then $W^\perp = \{v \in V : \lambda(v) = 0 \;\; \forall \lambda \in W\} \subset V$

is $\rho(\Gamma')$ - invariant and hence $W^{\perp} = (0)$ or V implying $W = V^*$ or (0^*). Next by Prop. 5.21 since ρ is s.i. proximal there exists $(\gamma_n)_{n\geq 1}$ in Γ such that $(\rho(\gamma_n))_{n\geq 1}$ is contracting, that is:

$$\frac{\rho(\gamma_n)}{\lambda_n} \longrightarrow u \,, \quad u \text{ of rank } 1 .$$

But then

$$\frac{\rho(\gamma_n)^*}{\lambda_n} \longrightarrow u^*$$

and since $\rho^*(\gamma_n^{-1}) = \rho(\gamma_n)^*$ this implies

$$\frac{\rho^*(\gamma_n^{-1})}{\lambda_n} \longrightarrow u^*$$

It is easy to verify that u^* is rank 1;

thus $(\rho^*(\gamma_n^{-1}))_{n\geq 1}$ is a contracting sequence and hence by Prop. 5.21, ρ^* is strongly irreducible proximal $\blacksquare$

Thus Thm 5.6 applied to ς and ς^* gives us Γ-equivariant measurable maps

$$\varphi : G/P \longrightarrow \mathbb{P}(V)$$

$$\varphi^* : G/P \longrightarrow \mathbb{P}(V^*).$$

and hence a Γ-equivariant measurable map

$$G/P \times G/P \longrightarrow \mathbb{P}(V) \times \mathbb{P}(V^*),$$

where now G and Γ act diagonally.

Next we need to know something about the G-orbits in $G/P \times G/P$. This goes under the name of Bruhat decomposition which is rather standard so we will be brief. Let as usual

$$A = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_i > 0, \ \prod_{i=1}^n \lambda_i = 1 \right\}$$

$$K = SO(n)$$

$$P = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} : \det(\ ) = 1 \right\}.$$

Let $W = N_K(A)/Z_K(A)$. Then by its very definition W acts faithfully by conjugation on A, in fact $\forall\, w \in W$, $\exists\, \sigma \in S_n$:

$$w \,\mathrm{diag}(\lambda_1, \dots, \lambda_n)\, w^{-1} = \mathrm{diag}(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$$

and the resulting homomorphism

$$W \longrightarrow S_n$$

is an isomorphism.

It is now usefull to thing of G/P as the set of complete flags in k^n where P is the stabilizer of the flag

$$(e_1) \subset (e_1, e_2) \subset \dots \subset (e_1, \dots, e_{n-1})$$

where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ and $(e_1, \dots, e_c)$ is the linear span of $e_1, \dots, e_c$. If $w_0 \in W$ is the element corresponding to the permutation

$\sigma_0 \in S_n$, $\quad \sigma_0(i) = n-i$, $\quad 1 \leq i \leq n$

then

$w_0 P w_0^{-1}$ is the stabilizer of the flag

$(e_n) \subset (e_n, e_{n-1}), \ldots \subset (e_n, \ldots, e_2)$

and hence

$$w_0 P w_0^{-1} = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} : \det(\ ) = 1 \right\}.$$

Thus $P \cap w_0 P w_0^{-1} \supset A$.

<u>Lemma 6.6.</u> The diagonal G-orbits in $G/P \times G/P$ are in 1-1 correspondence with W, in fact

$$G/P \times G/P = \bigsqcup_{w \in W} G(eP, wP)$$

and the G-orbit of $(eP, w_0 P)$ is open, dense and of full measure in $G/P \times G/P$.

With Corollary 4.27 Chapter 4 we deduce:

<u>Corollary 6.7.</u> The map: $G/A \longrightarrow G/P \times G/\underline{P}$

$$gA \longmapsto g(eP, w_0 \underline{P})$$

is G-equivariant with image a subset of full measure. The Γ-action on $G/P \times G/\underline{P}$ is ergodic.

<u>Proof:</u> The first part follows from lemma 6.6. The second part follows from the fact ~~that~~ A being closed non-compact, by Cor. 4.27, the Γ-action on G/A is ergodic, but then by the first part this implies that the Γ-action on $G/P \times G/P$ is ergodic. ∎

Now we turn to the definition of the third map. For every $v \in V \setminus (0)$ and $\lambda \in V^* \setminus (0)$

with $\lambda(v) \neq 0$, define:

$$\begin{aligned} V &\longrightarrow V \\ w &\longmapsto \frac{\lambda(w)\cdot v}{\lambda(v)} \end{aligned}$$

which is a rank 1 endomorphism with image $k.v$; in addition, and this is essential, it depends only on $\bar{v} \in \mathbb{P}(V)$ and $\bar{\lambda} \in \mathbb{P}(V^*)$ and we'll denote it by $P_{(\bar{v},\bar{\lambda})}$. It is an easy verification that the map:

$$\begin{aligned} \{(\bar{v},\bar{\lambda}) \in \mathbb{P}(V)\times\mathbb{P}(V^*) : \lambda(v)\neq 0\} &\longrightarrow \mathrm{End}(V) \\ (\bar{v},\bar{\lambda}) &\longmapsto P_{(\bar{v},\bar{\lambda})} \end{aligned}$$

is G-equivariant in the following sense:

$$P_{(g\bar{v},\, \overline{g^{*-1}\lambda})} = g\, P_{(\bar{v},\bar{\lambda})}\, g^{-1}, \quad g \in G.$$

In order to compose this map with $\varphi \times \varphi^*$

We show:

<u>Lemma 6.8.</u> The set $\{(x,y) \in G/P \times G/P :$

$\varphi(x) \not\subset \operatorname{Ker} \varphi^*(y)\}$ has full measure.

<u>Proof</u>: This set ~~[illegible]~~ is a Γ-invariant subset of $G/P \times G/P$ and hence (Cor. 6.7) by ergodicity it either has full measure or zero measure. Assume we are in the latter case, that is, for a.e. $(x,y) \in G/P \times G/P$ $\varphi(x) \subset \operatorname{Ker} \varphi^*(y)$. In particular, by Fubini, there exists $x \in G/P$ such that $\varphi(x) \subset \operatorname{Ker} \varphi^*(y)$ for a.e. $y \in G/P$. Since Γ is countable, the set of such y's can be taken Γ-invariant and still of full measure. Thus there is y such that $\varphi(x) \subset \operatorname{Ker} \varphi^*(\gamma y)$ $\forall \gamma \in \Gamma$. But: $\varphi^*(\gamma y) = \rho^*(\gamma)(\varphi^*(y))$ and hence

$\varphi(x) \subset \bigcap_{\gamma \in \Gamma} \rho^*(\gamma)(\operatorname{Ker} \varphi^*(y))$. This implies

$$0 \neq \bigcap_{\gamma \in \Gamma} \rho^*(\gamma)(\operatorname{Ker} \varphi^*(y)) \subsetneq V$$

and contradicts the irreducibility of ρ^*.

$\boxed{\S}$

Thus we can finally define

$$\cancel{E : G/A} \quad E : G/A \longrightarrow \operatorname{End}(V)$$

by $E(gA) = P_{\left(\varphi(gP),\, \varphi^*(g w_0 P)\right)}$

<u>Corollary 6.9.</u> The map E satisfies

(1) E is $A d \circ \rho$ - equivariant measurable

(2) for a.e. $x \in G/A$, $E(x) \in \operatorname{End}(V)$

has rank 1.

(3) The map E is not essentially constant.

Proof: We already showed (1) and (2).

Let's prove (3): if E where essentially constant then $E(x) = P$ for a.e. $x \in G/A$ and P is a rank 1 endomorphism. Now again the set of such x's can be taken Γ-invariant. Thus, for some $x \in G/A$ and every $\gamma \in \Gamma$, $E(\gamma x) = P$, implying by Γ-equivariance that $P = \rho(\gamma) P \rho(\gamma)^{-1}$ $\forall \gamma \in \Gamma$ and hence $\operatorname{Im} P$ is $\rho(\Gamma)$-invariant. But $\dim \operatorname{Im} P = 1$ and $\dim V \geq 2$ which contradicts the irreducibility of ρ. $\boxed{\text{≡}}$

6.2. Step 2, the Main Lemma.

Here we will consider the following general situation. We have a representation

$$\delta_\xi : \Gamma \longrightarrow GL(W)$$

where W is a f.d. k-vector space; in our application W will be $\mathrm{End}(V)$ and $\delta(\gamma)u := \rho(\gamma)u\,\rho(\gamma)^{-1}$. Let $F_\Gamma(G, W)$ denote the k-vector space of classes of measurable Γ-equivariant maps $G \to W$, where two maps that coincide almost everywhere are identified. Fix a probability measure $\mu \in M^1(G)$ that is equivalent to the Haar measure of G. We endow $F_\Gamma(G, W)$ with the topology having as subbasis of open sets:

$$V(f, \varepsilon, \delta) = \{ f' \in F_\Gamma(G, W) : \mu\{x \in G : \|f(x) - f'(x)\| > \varepsilon\} < \delta \}$$

where $f \in F_\Gamma(G, W)$, $\varepsilon > 0$, $\delta > 0$.
This is also called the topology of convergence in measure.

<u>Exercise</u>: Let $\mathcal{F}([0,1])$ be the vectorspace of ~~m~~ Lebesgue measurable functions $[0,1] \to \mathbb{R}$ with the topology of convergence in measure. Then $\mathcal{F}([0,1])$ is a topological vectorspace which is Hausdorff and does not admit any nonzero continuous linear functional.

Coming back to our situation, define $\forall g \in G$,

$$T(g)f(x) = f(xg), \quad f \in F_\Gamma(G, W).$$

Then $T : G \to GL(F_\Gamma(G,W))$ is a homomorphism and

Lemma 6.10.

(1) $F_\Gamma(G,W)$ is a topological vector space over k which is Hausdorff.

(2) For every $f \in F_\Gamma(G,W)$, the map

$$\begin{aligned} G &\longrightarrow F_\Gamma(G,W) \\ g &\longmapsto T(g)f \end{aligned}$$

is measurable.

We leave this as a tedious exercise.

The following is now the main Lemma:

<u>Lemma 6.11.</u> Let $S < A$ be a closed non-compact subgroup and $I \subset F_\Gamma(G, W)$ a finite dimensional $T(S)$-invariant subspace. Then

$$\mathrm{Lin}\,\{T(c)f : c \in Z_G(S), f \in I\}$$

is finite dimensional.

<u>Proof:</u> Since I is finite dimensional we can choose measurable maps $f_1, \dots, f_\ell : G \to W$, $\ell = \dim I$ such that the corresponding classes form a basis of I. This provides a linear section $I \xrightarrow{\sigma} \mathrm{Map}(G, W)$ into the space of measurable maps ~~on~~ $G \to W$ and hence every $x \in G$ gives rise to a well defined linear ~~[illegible]~~ map

$$\begin{aligned} I &\longrightarrow W \\ f &\longmapsto \sigma(f)(x) \end{aligned}$$

which we denote $f(x)$ by abuse of notation.

Given vector spaces B, C we let $\mathrm{Sym}^2(B, C)$ denote the vector space of quadratic maps $B \longrightarrow C$, that is, the restriction to the diagonal of bilinear symmetric maps $B \times B \longrightarrow C$.

Now we consider

$$q : G \longrightarrow \mathrm{Sym}^2(\mathcal{I}, \mathcal{I} \otimes W)$$
$$x \longmapsto \{ f \longmapsto f \otimes f(x) \}.$$

Now we establish some equivariance properties.

(1) For a.e. $x \in G$ and every $\gamma \in \Gamma$:

$$q(\gamma x)(f) = f \otimes f(\gamma x) = f \otimes \delta(\gamma)(f(x))$$
$$= (\underset{\mathcal{I}}{\mathrm{Id}} \otimes \delta(\gamma))(f \otimes f(x))$$
$$= (\mathrm{Id} \otimes \delta(\gamma))(q(x)(f)).$$

Thus if we define for $r \in Sym^2(I, I \otimes W)$:

$$(\tilde{\delta}(\gamma) r)(f) = Id \otimes \rho(\gamma)\,(r(f))$$

we obtain a representation $\tilde{\delta}$ of Γ in $Sym^2(I, I \otimes W)$ and:

$$q(\gamma x) = \tilde{\delta}(\gamma)(q(x)) \qquad (1)$$

for a.e. $x \in G$, $\forall \gamma \in \Gamma$.

(2) For every $s \in S$ and a.e. $x \in G$:

$$q(xs)(f) = f \otimes f(xs) = f \otimes (T(s) f)(x)$$

$$= (T(s^{-1}) \otimes Id_W)\,(\underbrace{T(s)f \otimes (T(s)f)(x)}_{q(x)(T(s)f)})$$

Thus defining for $r \in Sym^2(I, I \otimes W)$

$$(\tilde{T}(s) r)(f) = T(s^{-1}) \otimes Id\,(r(T(s)f))$$

we obtain a representation $\tilde{T}$ of S in $Sym^2(I, I \otimes W)$ and:

$$q(xs) = \tilde{T}(s)\,(q(x)) \qquad (2)$$

for every $s \in S$ and almost every $x \in G$.

Now consider for every $x \in G$ the map

$$q_x : Z_G(S) \longrightarrow \mathrm{Sym}^2(I, I \otimes W)$$
$$c \longmapsto q(xc)$$

and let S_x be the linear span of the essential image of q_x.

Then we have: for a.e. $x \in G$ and every $\gamma \in \Gamma$, $\cancel{q}\ S_{\gamma x} = \tilde{\rho}(\gamma)(S_x)$

and for every $s \in S$ and a.e. $x \in G$:

$$S_{xs} = S_x .$$

Hence the function $G/_S \longrightarrow \mathbb{N}$

$$x \longmapsto \dim S_x$$

being measurable Γ-invariant, is essentially constant; remember that $S \le G$ being closed non-compact, Γ acts ergodically on G/S.

Let $N \in \mathbb{N}$ be this constant.

Claim: there is $J \subset Z_G(S)^N$ of positive measure such that for all

$(c_1, .., c_N) \in J$ and almost every $x \in G$:

$$\mathrm{Lin}\{q_x(c_1), .., q_x(c_N)\} = S_x .$$

Indeed, otherwise for a.e. $(c_1, .., c_N) \in Z_G(S)^N$ and a.e. $x \in G$

$$\dim \mathrm{Lin}\{q_x(c_1), .., q_x(c_N)\} \le N-1 .$$

and by Fubini this implies that for a.e. $x \in G$ and a.e. $(c_1, .., c_N) \in Z_G(S)^N$

$$\dim \mathrm{Lin}\{q_x(c_1), .., q_x(c_N)\} \le N-1 .$$

But this contradicts the definition of S_x and the fact that $\dim S_x = N$ for a.e. $x \in G$.

Fix now $c_1, \dots, c_N$ in $\mathcal{Z}_G(S)$ such that

$$\mathrm{Lin}\{q_x(c_1), \dots, q_x(c_N)\} = S_x$$

for a.e. $x \in G$.

Thus for a.e. $x \in G$ and a.e. $c \in \mathcal{Z}_G(S)$:

$$q(xc) = \sum_{i=1}^{N} \alpha_i(x, c)\, q(xc_i)$$

where $\alpha : G \times \mathcal{Z}_G(S) \longrightarrow k$ are measurable functions. We proceed to establish some inversance properties of α.

First from (1) that is, $q(\gamma x c) = \tilde{\delta}(\gamma) q(xc)$

we get:

$$\sum \alpha_i(\gamma x, c)\, q(\gamma x c_i) = \tilde{\delta}(\gamma) \left(\sum \alpha_i(x, c)\, q(x c_i)\right)$$

$$= \sum \alpha_i(x, c)\, q(\gamma x c_i)$$

and hence $\alpha_i(\gamma x, c) = \alpha_i(x, c)$

for all $\gamma \in \Gamma$ and a.e. $(x, c) \in G \times \mathcal{Z}_G(S)$.

Next we have for a.e. $x \in G$, $s \in S$, $c \in Z_G(S)$:

$$q(xsc) \overset{!}{=} q(xcs) = \tilde{T}(s)(q(xc)).$$

Hence:

$$\sum \alpha_i(xs,c)\, q(xsc_i) = \sum \alpha_i(x,c)\, \underbrace{\tilde{T}(s)(q(xc_i))}_{q(xsc_i)}$$

hence $\alpha_i(xs,c) = \alpha_i(x,c)$.

Together with the ergodicity of Γ on G/S this implies that $\alpha_i(x,c) = \alpha_i(c)$ for a.e. $x \in G$, for a.e. $c \in Z_G(S)$.

Thus: $q(xc) = \sum \alpha_i(c)\, q(xc_i)$

that is $f \otimes f(xc) = \sum \alpha_i(c)\, f \otimes f(xc_i)$

$$= f \otimes \sum_{i=1}^{N} \alpha_i(c)\, f(xc_i)$$

Which implies: $f(xc) = \sum_{i=1}^{N} \alpha_i(c)\, f(xc_i)$

Thus if $f_1, \dots, f_\ell$ is a basis of I

then $\{ T(c_i) f_j : 1 \leq i \leq N, 1 \leq j \leq \ell \}$

is a basis of $\mathrm{Lin}\{ T(c) f : c \in Z_G(S), f \in I \}$.

$\boxed{\mathrm{WW}}$

6.3. Step 3 and 4

As announced, the next step involves using the assumption $\mathrm{rank}_{\mathbb{R}} G \geq 2$:

Prop. 6.12. Assume $n \geq 3$. Then there are closed non-compact subgroups $S_1, \dots, S_\ell$ in A such that

$$SL(n, \mathbb{R}) = Z_G(S_1) \cdots Z_G(S_\ell)$$

The $S_1, \dots, S_\ell$ come possibly with repetition.

The proof goes by induction on $n \geq 3$. We illustrate the case $n = 3$:

Let

$$S_1 = \left\{ \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda^{-2} \end{pmatrix} : \lambda > 0 \right\}$$

$$S_2 = \left\{ \begin{pmatrix} \lambda & & \\ & \lambda^{-2} & \\ & & \lambda \end{pmatrix} : \lambda > 0 \right\}$$

$$S_3 = \left\{ \begin{pmatrix} \lambda^{-2} & & \\ & \lambda & \\ & & \lambda \end{pmatrix} : \lambda > 0 \right\}$$

Then

$$Z_G(S_1) = \left\{ \begin{pmatrix} A & 0 \\ 0 & b \end{pmatrix} : \begin{array}{l} A \in GL(2, \mathbb{R}) \\ (\det A) \cdot b = 1 \end{array} \right\}$$

$$Z_G(S_2) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 3 & 0 \\ c & 0 & d \end{pmatrix} : (ad - bc)\, 3 = 1 \right\}$$

$$Z_6(s_3) = \left\{ \begin{pmatrix} b & 0 \\ 0 & A \end{pmatrix} : b \cdot \det A = 1 \right\}.$$

Then the product $Z_6(s_1) Z_6(s_2) Z_6(s_3)$ clearly contains

$$\underline{P} = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} : \det(\ ) = 1 \right\}.$$

Next: $Z_6(s_1) \ni \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix}$

$$Z_6(s_2) \ni \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$Z_6(s_3) \ni \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and hence $Z_6(s_1) Z_6(s_2) Z_6(s_3)$ contains a set W' of representatives of the

Weyl group $N_K(A)/Z_K(A)$. The Bruhat decomp. says that $SL(3,\mathbb{R}) = P W' P$ which implies that Prop. 6.12 holds with the sequence $s_1, s_2, s_3, s_1, s_2, s_3, s_1, s_2, s_3$.

Now we apply Step 2 to the context of Thm 6.1 in particular $W = End(V)$ and $\delta = Ad \circ \rho$. We start with Corollary 6.9 and consider the map E as a $T(A)$-inv. vector in $F_\Gamma(G, End(V))$. We now use Step 2 and 3 to prove:

Prop. 6.13. Assume $n \geq 3$. Then

$$I := Lin\{T(g)E : g \in G\}$$

is a finite dimensional subspace of $F_\Gamma(G, End(V))$.

Proof: Take $S_1, \dots, S_e$ as in lemma 6.12 and define $I_0 = k \cdot E$, and for $j \geq 1$

$$I_j = \{ T(c) f : c \in Z_G(S_j) , f \in I_{j-1} \}.$$

Observe that by prop. 6.12:

$I_e = I$ and we proceed to show by recurrence that $\dim I_j < +\infty$. By definition $\dim I_0 = 1$. Let's assume $j \geq 1$ and $\dim I_{j-1} < +\infty$. Now I_{j-1} is $T(Z_G(S_{j-1}))$ invariant by construction; since

$$Z_G(S_{j-1}) \supset A \supset S_j$$

the subspace I_{j-1} is $T(S_j)$-invariant finite dimensional and hence by lemma 6.11

$$I_j = \{ T(c) f : c \in Z_G(S_j) , f \in I_{j-1} \}$$

is finite dimensional. $\boxed{\xi}$

6.4. <u>Step 5.</u>

Let $I = \mathrm{Lin}\{T(g)E : g \in G\}$ and $f_1, \ldots, f_N$ be a basis of this k-vector-space. Since I is a finite dimensional Hausdorff topological vector space the map

$$\begin{aligned} k^N &\longrightarrow I \\ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{pmatrix} &\longmapsto \sum_{i=1}^{N} \lambda_i f_i \end{aligned}$$

is a topological vectorspace isomorphism. In particular if $m(g)$ is the matrix of $T(g)$ in this basis it follows from lemma 6.10 that the resulting homomorphism $m : G \to GL(n, k)$ is measurable.

Now we have:

Thm 6.14. (Mackey) Let G be l.c.s.c. H a second countable topological group and $m: G \to H$ a measurable homomorphism. Then m is continuous.

Proof: Let μ be the Haar measure on G. First we show that if $M \subset G$ is measurable if $\mu(M) > 0$ then $M \cdot M^{-1} \ni e$ is a neighborhood of e. Pick $K \subset M$ compact with $\mu(K) > 0$. Let $W \supset K$ be open such that $\mu(W) < 2\mu(K)$. Since K is compact we can find $N \ni e$ neighboorhood of e such that $N = N^{-1}$ and $N \cdot K \subset W$. So $x \cdot K \subset W$ $\forall x \in N$: since $\mu(xK) = \mu(K)$ and $\mu(W) < 2\mu(K)$

we must have $xK \cap K \neq \phi$ $\forall x \in N$ hence $N \subset K \cdot K^{-1}$!

Now we prove the theorem. We can assume that $m(G) = H$. Let $U \ni e$ be a neighborhood of e and $V \ni e$ symmetric open with $V^2 \subset U$. Let $\{y_n : n \in \mathbb{N}\}$ be a countable dense subset of H and $\{g_n : n \in \mathbb{N}\} \subset G$ with $m(g_n) = y_n$. Then $H = \bigcup_{n \geq 1} y_n \cdot V$ hence

$G = \bigcup_{n \geq 1} g_n m^{-1}(V)$. Thus for some n_0, $\mu(g_{n_0} m^{-1}(V)) > 0$ hence $\mu(m^{-1}(V)) > 0$.

This implies by the above that

$$m^{-1}(U) \supset m^{-1}(V) \cdot m^{-1}(V)$$

is a neighborhood of e. $\boxed{}$

Let now $C_\Gamma(G, \mathrm{End}(V))$ be the vector-space of continuous $\mathrm{Ad}\circ g$-Γ-equivariant maps; we ~~still~~ denote by $\tilde{T}(g)$ the linear map $\tilde{T}(g)f(h) = f(hg)$, $f \in C_\Gamma(G, \mathrm{End}\, V)$.

<u>Corollary 6.15.</u> The map E coincides almost everywhere with a unique continuous map ~~E~~ $\tilde{E} \in C_\Gamma(G, \mathrm{End}(V))$ and $k = \mathbb{R}$ or $\mathbb{C}$. The vector space

$$\tilde{I} = \{\tilde{T}(g)\tilde{E} : g \in G\}$$

is isomorphic to I via the map which to a ~~formal~~ continuous $f \in C_\Gamma(G, \mathrm{End}\, V)$ associates its class in $F_\Gamma(G, \mathrm{End}\, V)$ and this map intertwines $\tilde{T}$ and T.

<u>Proof:</u> The main point is the first assertion. The rest follows from the fact that