that if $f_1, f_2 \in C_\Gamma(G, \mathrm{End}\,V)$ coincide almost everywhere, then they coincide.

Coming back to the beginning of 6.4 we now know that $m : G \to GL(N, k)$ is continuous. For every $g \in G$ there is $c_1(g), \ldots, c_N(g) \in k$ with

$$T(g)E = \sum_{i=1}^{N} c_i(g) f_i$$

and since m is continuous, the functions $c_1, \ldots, c_N$ are continuous. Thus $\forall g \in G$ and almost every $h \in G$:

$$T(g)E(h) = \sum c_i(g) f_i(h)$$

hence by Fubini: there exists $h \in G$ such that $\cancel{\Sigma}\ E(hg) = \sum c_i(g) f_i(h)$ for almost every $g \in G$. This implies that E coincides almost everywhere with a (unique) continuous map $\widetilde{E} \in C_{\Gamma}(G, \operatorname{End} V)$.

Since E is not essentially constant, $\widetilde{E}$ is not constant. But G is connected; if k is a finite extension of $\mathbb{Q}_p$ then $\operatorname{End} V$ is totally disconnected, hence $k = \mathbb{R}$ or $\mathbb{C}$. $\boxed{\xi}$

6.5. Step 6. The proof of Thm 6.1.

Let as above $\widetilde{I} = \operatorname{Lin}\{\widetilde{T}(g)\widetilde{E} : g \in G\} \subset C_{\Gamma}(G, \operatorname{End} V)$.

Consider
$$\mathcal{E} : \widetilde{I} \longrightarrow \operatorname{End}(V)$$
$$f \longmapsto f(e)$$

$$(1) \quad \mathcal{E}\,\widetilde{T}(\gamma) = \operatorname{Ad} \rho(\gamma)\, \mathcal{E} \quad \forall \gamma \in \Gamma.$$

Indeed:
$$\mathcal{E}\,\widetilde{T}(\gamma)(f) = f(\gamma) = f(\gamma \cdot e)$$
$$= \operatorname{Ad} \rho(\gamma)\,(f(e))$$
$$= \operatorname{Ad} \rho(\gamma)\, \mathcal{E}(f).$$

Now
$$G \longrightarrow GL(\tilde{I})$$
$$g \longmapsto \tilde{T}(g)|_{\tilde{I}}$$

is a continuous finite dimensional representation of $G = SL(n, \mathbb{R})$. But $SL(n, \mathbb{R})$ being simple real algebraic, this representation in any basis of $\tilde{I}$ is given by matrices that are polynomials in the entries of g.

The kernel $\operatorname{Ker} \mathcal{E}$ is because of (1) $\tilde{T}(\Gamma)$-invariant and since Γ is Zariski dense in G (Thm 6.4), $\operatorname{Ker} \mathcal{E}$ is $\tilde{T}(G)$-invariant. But then if $f \in \operatorname{Ker} \mathcal{E}$, $\tilde{T}(g) f \in \operatorname{Ker} \mathcal{E}$ $\forall g \in G$, thus $f(g) = 0$ $\forall g \in G$ hence $\operatorname{Ker} \mathcal{E} = (0)$. This concludes the proof of Thm 6.1. $\boxed{\xi}$