Example 1.7. $G = SO(n,1)^{\circ}$

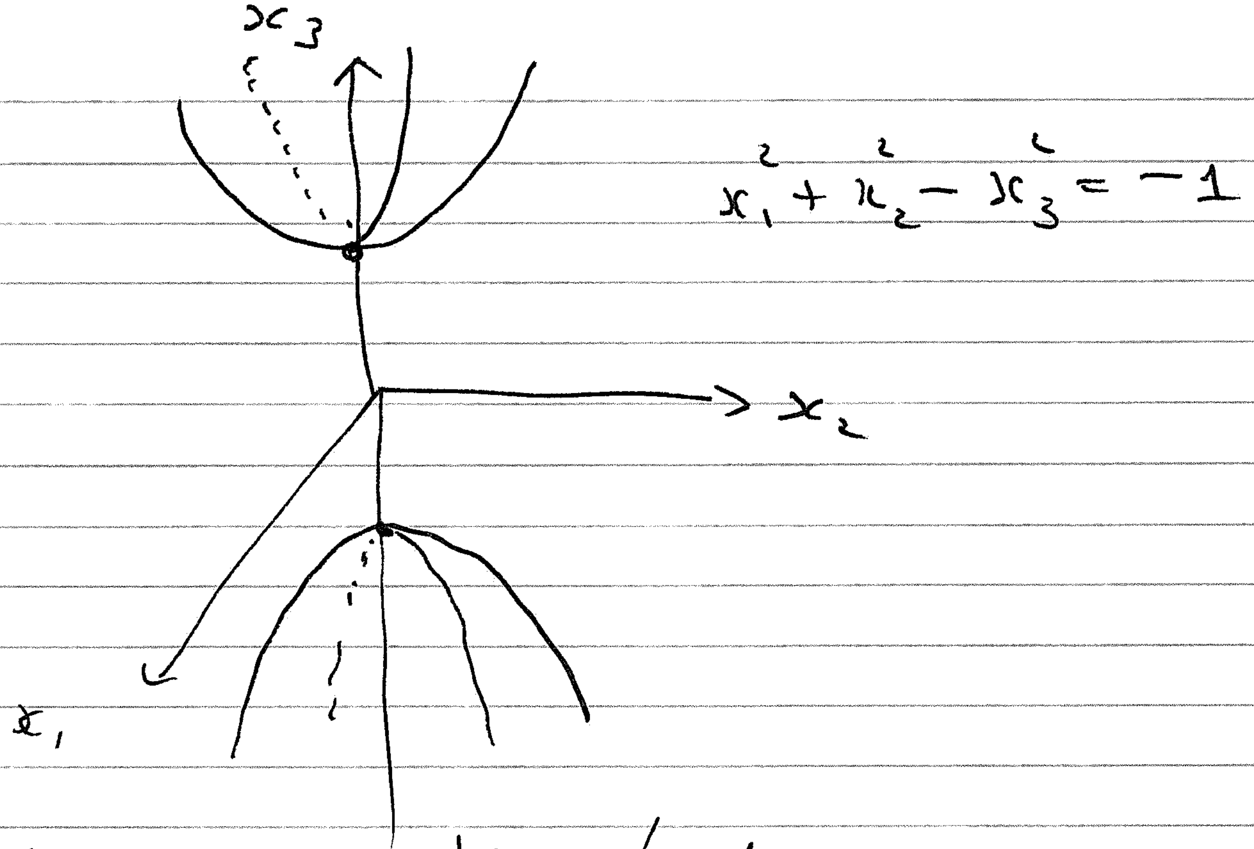
For every dimension $n \geq 2$ there is up to isometry exactly one simply connected, complete riemannian manifold of constant sectional curvature -1; it is ~~a~~ called real hyperbolic n-space $\mathbb{H}^n_{\mathbb{R}}$ and here is a description of it.

On $\mathbb{R}^{n+1}$ consider the form of signature $(n,1)$:

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$$

The hyperboloïd

$$\{ x \in \mathbb{R}^{n+1} : \langle x, x \rangle = -1 \}$$



has two components. Let

$$\mathbb{H}^n_{\mathbb{R}} = \left\{ x \in \mathbb{R}^{n+1} : \langle x, x \rangle = -1 \text{ and } x_{n+1} > 0 \right\}$$

Then for $x \in \mathbb{H}^n_{\mathbb{R}}$

$$T_x \mathbb{H}^n_{\mathbb{R}} = \left\{ y \in \mathbb{R}^{n+1} : \langle y, x \rangle = 0 \right\}$$

Then the restriction of $\langle \, , \, \rangle$ to $T_x \mathbb{H}^n_{\mathbb{R}}$ is positive definite and gives a riemannian metric on $\mathbb{H}^n_{\mathbb{R}}$ that has constant sectional curvature -1.

Its group of orientation preserving isometries can be identified with $SO(n,1)^\circ$, the connected component of the identity of

$$SO(n,1) = \{ g \in SL(n+1, \mathbb{R}) : \langle gx, gy \rangle = \langle x, y \rangle \ \forall x, y \in \mathbb{R}^{n+1} \}.$$

This identification is via:

$$SO(n,1)^\circ \longrightarrow Is(\mathbb{H}^n_{\mathbb{R}})^\circ$$

$$g \longmapsto g|_{\mathbb{H}^n_{\mathbb{R}}}.$$

For $n = 2$, we recover $\mathbb{H}^2_{\mathbb{R}}$, a different model for the Poincaré half-plane; any isometry $\mathbb{H}^2_{\mathbb{R}} \longrightarrow \mathbb{H}^2$ induces an isomorphism $SO(2,1)^\circ \longrightarrow PSL(2,\mathbb{R})$.

We have also seen that if one fixes the diffeo type of a compact quotient of $\mathbb{H}^2_{\mathbb{R}}$ then there is a $(6g-6)$-parameter family of cocompact lattices in $SO(2,1)^\circ$ giving diffeomorphic surfaces.

In contrast:

<u>Theorem 1.8</u> (Mostow)

(geometric form) Let M_1, M_2 be compact riemannian manifolds with a constant sectional curvature -1 metric. Then any homotopy equivalence $f: M_1 \to M_2$ is homotopic to an isometry.

(group theoretic form) ~~Let~~ $\cancel{\Gamma_1 \to \Gamma_2}$

Let Γ_1, Γ_2 be discrete cocompact subgroups of $SO(n,1)^\circ$. Then any group isomorphism $\theta: \Gamma_1 \to \Gamma_2$

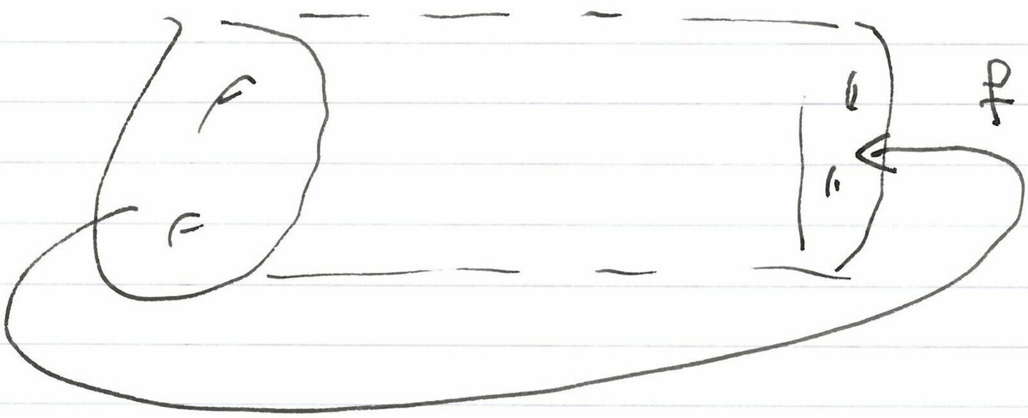
extends to a Lie group automorphism

$$\Theta_{ext} : SO(n,1)^{\circ} \circlearrowleft .$$

Thus a compact smooth manifold M with $\dim M \geq 3$ admits at most one ~~cst hyperbol~~ riemannian metric with constant sectional curvature -1, and the question is to classify up to diffeomorphisms those who admit one. As a result of efforts of many mathematicians (see N. Bergeron, Séminaire Bourbaki, n° 1078, Janvier 2014), the problem is solved in dimension 3. The end result is relatively simple to describe. Let Σ_g, $g \geq 2$ as before and $f : \Sigma_g \circlearrowleft$ a diffeomorphism.

Then the mapping torus $M_{g,f}$ is the compact 3-manifold defined as:

$$\Sigma_g \times [0,1] \Big/ (x,0)\sim(f(x),1) .$$



Then

Thm 1.9 (Thurston-Sullivan) $M_{g,f}$ admits a r.m. of constant sect. curvature -1 $\iff$ f is pseudo-Anosov.

Previously Thurston had classified diffeomorphisms of surfaces with the major tool, the Thurston boundary of Teichmüller space: pseudo-Anosov are in a sense the most "hyperbolic" diffeomorphisms.

Then came Agol, Wise, Kahn–Markovic etc. —

<u>Thm 1.10.</u> Let M_0 be a compact 3-manifold admitting a r.m. of sect. curv. $= -1$. Then there exists a finite covering $M \to M_0$ such that $M \cong M_{g,f}$ for some $g \geq 2$ and pseudo-Anosov f.

<u>Example 1.11.</u> $G = SL(n,\mathbb{R})$, then $\Gamma = SL(n,\mathbb{Z})$ is a lattice in $SL(n,\mathbb{R})$ as we will see in chapter 3. In this situation it is easy to describe the invariant measure on the manifold G/Γ. Let ~~we~~ $d = \dim G = n^2 - 1$ and $\omega \in \Omega^d(G)$

the essentially unique left invariant volume form on G; ~~since~~ one verifies easily that it is right invariant as well. Thus via the covering map

$$\pi : G \longrightarrow G/\Gamma$$

it induces a G-invariant volume form Ω on G/Γ. The theorem is then that

$$\int_{G/\Gamma} \Omega < +\infty .$$

This situation is especially interesting as we'll see that $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ is not compact. Many properties of this homogeneous space are based on its identification with the space

$$\mathcal{R}^{(1)} = \left\{ \Lambda < \mathbb{R}^n : \Lambda \text{ is a lattice with } \operatorname{vol}(\Lambda \backslash \mathbb{R}^n) = 1 \right\}$$

Let's just say for the sake of concreteness that if

$$\Gamma = \mathbb{Z}b_1 + \dots + \mathbb{Z}b_n$$

is a lattice in $\mathbb{R}^n$,

$$\mathrm{Vol}\left(\Gamma\backslash\mathbb{R}^n\right) = \left|\det(b_1, \dots, b_n)\right|.$$

Then $SL(n,\mathbb{R})$ acts on $\mathcal{R}^{(1)}$ by

$$SL(n,\mathbb{R}) \times \mathcal{R}^{(1)} \longrightarrow \mathcal{R}^{(1)}$$
$$(g, \Lambda) \longmapsto g(\Lambda)$$

and it follows from the discussion in Example 1.5 that this action is transitive. The stabilizer of $\mathbb{Z}^n$ is then $SL(n,\mathbb{Z})$.

The group $SL(n,\mathbb{Z})$ is an example of arithmetic lattice in $SL(n,\mathbb{R})$; to reach the general notion of arithmetic lattice

We'll need the notion of linear algebraic group and field of definition. Roughly a linear algebraic group is a subgroup of some $GL(n, \mathbb{C})$ defined by polynomial equations.

<u>Def 1.12</u>

(1) A linear algebraic group is a subgroup $\mathbb{G} < GL(n, \mathbb{C})$ which is the set of common zeros of a set of polynomials in $\mathbb{C}\left[X_{ij}, (\det X_{ij})^{-1}\right]$.

(2) Given a subring $A \subset \mathbb{C}$ we set

~~$\mathbb{G}(A) = \{ g$~~

$$GL(n, A) = \{ g \in GL(n, \mathbb{C}) : g_{ij} \in A$$
$$(\det g_{ij})^{-1} \in A \}$$

which is a subgroup of $GL(n, \mathbb{C})$ and the group of A-points of $\mathbb{G}$ is:

$$\mathbb{G}(A) := \mathbb{G} \cap GL(n, A).$$

(3) We say that $\mathbb{G}$ is defined over a subfield $k \subset \mathbb{C}$ if $\mathbb{G}$ is the set of common zeroes of polynomials in

$$k\left[x_{ij}, (\det x_{ij})^{-1}\right].$$

Example 1.13

Let $F: \mathbb{C}^2 \to \mathbb{C}$ be the quadratic form $F\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + \sqrt{2}\, y^2$. Then setting

$$J = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

We have

$$SO(F) = \{ g \in GL(2, \mathbb{C}) : \det g = 1$$

$$\text{and } {}^t g J g = J \}$$

so that $SO(F)$ is defined over $\mathbb{Q}(\sqrt{2})$.

We have $SO(F)(\mathbb{R}) \simeq SO(2,\mathbb{R})$ since over $\mathbb{R}$ the quadratic forms $x^2 + \sqrt{2}\,y^2$ and x^2+y^2 are equivalent. Observe that $SO(F)(\mathbb{Q}) = \pm Id$ while $SO(2,\mathbb{Q})$ is dense in $SO(2,\mathbb{R})$.

The fundamental theorem is due to Borel and Harish-Chandra:

Thm 1.14 Let $\mathbb{G} < GL(n,\mathbb{C})$ be a semisimple linear algebraic group defined over $\mathbb{Q}$. Then $\mathbb{G}(\mathbb{Z})$ is a lattice in $\mathbb{G}(\mathbb{R})$.

We will explain a little later what semisimple means; for the moment I just want to illustrate the power of this construction in the following:

Example 1.15

Let $F : \mathbb{C}^3 \longrightarrow \mathbb{C}$, $F_p\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + x_2^2 - p x_3^2$

with $p \geq 1$ an <u>integer</u>. Then

$$SO(F_p) = \{ g \in GL(3, \mathbb{C}) : {}^t g S_p g = S_p,$$

$$\det g = 1 \}$$

where $S_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -p \end{pmatrix}$ is a linear algebraic group defined over $\mathbb{Q}$, satisfying the hypothesis of the B-H.C. theorem.

Let's compute its group of real points: over $\mathbb{R}$ the quadratic forms F_p and F_1 are equivalent and $SO(F_p)(\mathbb{R})$ is conjugate to $SO(F_1)(\mathbb{R}) = SO(2,1)$ in $GL(3, \mathbb{R})$:

Thus we have an isomorphism of Lie groups:

$$j_p : SO(F_p)(\mathbb{R}) \longrightarrow SO(2,1)$$
$$g \longmapsto S_{\sqrt{p}}\, g\, S_{\sqrt{p}}^{-1}.$$

Now according to B-HC, $SO(F_p)(\mathbb{Z})$

$$= \{ g \in SL(3,\mathbb{Z}) : {}^t g\, S_p\, g = S_p \}$$

is a lattice in $SO(F_p)(\mathbb{R})$ and as a result, $j_p(SO(F_p)(\mathbb{Z}))$ is a lattice in $SO(2,1)$.

We will see that when p runs through the primes $\equiv 3 \pmod 4$ we get cocompact lattices and in addition they are non-conjugate; they are even unrelated in a stronger way that we will introduce shortly.

<u>Exercise</u>: What happens if p is a prime $\equiv 1 \pmod 4$?

There is an arithmetic way to construct many finite index subgroups of $GL(n,\mathbb{Z})$, namely for every $m \in \mathbb{N}$, $m \geq 1$:

$$\Gamma(m) := \{ g \in GL(n,\mathbb{Z}) : g \equiv \mathrm{Id} \mod m \}.$$

Thus if $\mathbb{G} < GL(n,\mathbb{C})$ is a linear algebraic group $\mathbb{G}(\mathbb{Z}) \cap \Gamma(m)$ is of finite index in $\mathbb{G}(\mathbb{Z})$ $\forall m \geq 1$. If $\mathbb{G}(\mathbb{Z})$ is a lattice in $\mathbb{G}(\mathbb{R})$ then all these finite index subgroup are lattices as well.

<u>Def 1.16.</u> Two subgroups Γ and Γ' of a group G are called commensurable if $\Gamma \cap \Gamma'$ is of finite index both in Γ and Γ'.

The fact that a finite index subgroup of a lattice is a lattice is a special case of:

<u>Lemma 1.17</u> Let $\Gamma < G$ be a lattice in a l.c. group G, and $\Gamma' < G$ a subgroup s.t Γ and Γ' are commensurable. Then Γ' is a lattice in G.

<u>Proof:</u>

(1) Since $\Gamma > \Gamma \cap \Gamma'$ is discrete so is $\Gamma \cap \Gamma'$. Now $\Gamma' = \bigcup_{i=1}^{l} \gamma_i (\Gamma' \cap \Gamma)$ ~~being~~ a finite union of discrete subsets is

discrete as well.

(2) Consider the diagramme of G-spaces:

$$\begin{tikzcd} & G/_{\Gamma\cap\Gamma'} \arrow[dl, "\gamma_\Gamma"'] \arrow[dr, "p_{\Gamma'}"] & \\ G/_{\Gamma} & & G/_{\Gamma'} \end{tikzcd}$$

If μ is a G-invariant probability measure on $G/_\Gamma$, then $\forall f \in C_b(G/_{\Gamma\cap\Gamma'})$ continuous bounded function,

$$\sum_{\gamma \in \Gamma/_{\Gamma\cap\Gamma'}} f(g\dot{\gamma})$$

is continuous bounded as well. Then

$$C_b(G/_{\Gamma\cap\Gamma'}) \longrightarrow \mathbb{R}$$

$$f \longmapsto \int_{G/_\Gamma} \sum_{\gamma\in\Gamma/_{\Gamma\cap\Gamma'}} f(g\dot{\gamma})\, d\mu(g)$$

define a positive G-invariant functional on $C_b(G/_{\Gamma\cap\Gamma'})$ and hence $\Gamma\cap\Gamma'$ is a lattice in G. Finally, observe that if ν is a G-invariant probability measure on $G/_{\Gamma\cap\Gamma'}$, then its push-forward $p_{\Gamma' *}(\nu)$ is a G-invariant probability measure on $G/_{\Gamma'}$. $\boxed{S}$

There is another simple mechanism by which one obtains new lattices:

<u>Lemma 1.18</u> Let $p: H \longrightarrow G$ be a continuous surjective homomorphism of second countable l.c. groups and $\Gamma < H$ a lattice. If $\operatorname{Ker} p$ is compact, then $p(\Gamma)$ is a lattice in G.

<u>Remark 1.19</u>: $\mathrm{Ker}\,p$ garanties that $p(\Gamma)$ is discrete. Indeed consider $H = \mathbb{R}^2$, $G = \mathbb{R}$, $p\begin{pmatrix} x \\ y \end{pmatrix} = x + \sqrt{2}\,y$. Then $p(\mathbb{Z}^2)$ is dense in $\mathbb{R}$.

<u>Proof</u>: From the assumption that G, H are l.c. second countable we deduce that p induces an isomorphism of l.c. groups $H/\mathrm{Ker}\,p \xrightarrow{\sim} G$ so that we may assume $G = H/\mathrm{Ker}\,p$. Let $V \subset H$ be an open neighborhood of $e \in H$ with $\overline{V}$ compact. Then $\overline{V} \cdot \mathrm{Ker}\,p$ is compact and $\overline{V} \cdot \mathrm{Ker}\,p \cap \Gamma$ is finite. Thus $p(V) \cap p(\Gamma)$ is finite. Since p is an open map, and G is Hausdorff this implies $p(\Gamma)$ is discrete.

Thus $G/_{p(\Gamma)}$ is a l.c. G-space and p induces a continuous, surjective, equivariant map:

$$\varphi : H/_{\Gamma} \longrightarrow G/_{p(\Gamma)}$$
$$h\cdot\Gamma \longmapsto p(h)\,p(\Gamma)$$

that is $\varphi(hx) = p(h)\,\varphi(x)$ $\forall h \in H$ $\forall x \in H/_{\Gamma}$.

Thus if α is an H-invariant probability measure on $H/_{\Gamma}$, $\varphi_*(\alpha)$ is a G-invariant probability measure on $G/p(\Gamma)$.

$\blacksquare$

We are almost ready to define the general notion of lattice. But before this I want to recall a basic notion of Lie theory.

<u>Def. 1.20.</u> (1) A Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is simple if it is non-abelian and

admits no other ideals than (0) and $\mathfrak{g}$.

(2) A Lie algebra over $\mathbb{F}$ is semisimple if it is a sum of simple ideals.

(3) A real or complex Lie group is semisimple if its Lie algebra is.

For instance a linear algebraic group $G \subset GL(n, \mathbb{C})$ is a complex Lie group and the hypothesis of the Borel-Harish Chandra theorem is that its Lie algebra is semisimple.

In the sequel "Lie group" will always refer to "real Lie group". We will be interest in lattices in simple and semisimple Lie groups.

Def. 1.21. Let G be a connected semisimple Lie group. A lattice $\Gamma < G$ is arithmetic, if there is a semisimple linear algebraic group $\mathbb{H}$ ~~[illegible]~~ defined over $\mathbb{Q}$ and a continuous surjective homomorphism

$$p : \mathbb{H}(\mathbb{R})^\circ \longrightarrow G$$

with compact kernel, such that $p(\mathbb{H}(\mathbb{Z}) \cap \mathbb{H}(\mathbb{R})^\circ)$ and Γ are commensurable.

To which extent all lattices in a simple Lie group are arithmetic is known except for a certain family; more on this later.

The Margulis arithmeticity theorem

makes a sharp distinction between simple Lie groups of real rank 1 and those of real rank $\geqslant 2$. Here is a notion of rank in a degree of generality sufficient for us:

<u>Definition 1.22</u>: The real rank of a semisimple Lie subgroup $G < GL(n, \mathbb{R})$ is the maximal dimension of a connected abelian subgroup of G consisting of matrices diagonalizable over $\mathbb{R}$. Such a subgroup is then simply isomorphic to $(\mathbb{R}^\times)^{\circ\, r}$ where $r = \operatorname{rank} G$.

<u>Examples 1.23</u>

(1) rank 0 : $SO(n)$, $n \geqslant 3$; $SU(n)$, $n \geqslant 2$.

(2) rank 1 : $SO(n,1)$, $n \geqslant 2$; $SU(n,1)$, $n \geqslant 2$; $Sp(n,1)$, $n \geqslant 2$; $F_{4(-20)}$.

In the same way $SO(n,1)$ is related to real hyperbolic ~~space~~ space $\mathbb{H}^n_{\mathbb{R}}$, $SU(n,1)$ is to complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, $Sp(n,1)$ to quaternionic $\mathbb{H}^n_{\mathbb{K}}$ and $F_{4(-20)}$ to the octonion plane $\mathbb{H}^2_{\mathbb{O}}$.

(3) rank n : $SL(n,\mathbb{R})$, $Sp(2n,\mathbb{R})$, $SU(p,n)$ $p \geq n$, $SO(p,n)$ $p \geq n$.

Actually the list in (2) is, up to local isomorphism, a complete list of the simple rank 1 Lie groups. (*)

Now we can complete the picture of what is known concerning arithmeticity:

(α)

The fundamental result of Margulis

is:

<u>Thm A</u>. Let $G \leq GL(n, \mathbb{R})$ be a connected simple Lie group of rank ≥ 2. Then all lattices in G are arithmetic.

Thm 1.24

(1) For every $n \geq 2$ there are non-arithmetic lattices in $SO(n,1)$. (Gromov-PS ~86)
and others for small n.

(2) There are non-arithmetic lattices in $SU(2,1)$, $SU(3,1)$ (Mostow ~80)

(3) All lattices in $Sp(n,1)$, $n \geq 2$ and $F_{4(-20)}$ are arithmetic. (Corlette ~91)

Concerning non-arithmeticity, there is a striking result of A. Borel in the case $PSL(2,\mathbb{R})$

Thm 1.25 Under the bijection

$$\mathcal{T}_g = \mathrm{Hyp}(\Sigma_g) \Big/ \mathrm{Diff}^+_0(\Sigma_g) \xrightarrow{\sim} \mathrm{Hom}_{hyp}(\pi_1(\Sigma_g), PSL(2,\mathbb{R})) \Big/ PSL(2,\mathbb{R})$$

there are for each $g \geq 2$ only finitely
up to the action of the mapping class group