many points in $\mathcal{T}_g$ that correspond to ~~...~~ conjugacy classes of homomorphisms $\pi_1(\Sigma_g) \to PSL(2,\mathbb{R})$ whose image in $PSL(2,\mathbb{R})$ is an arithmetic lattice.

Observe that all lattices coming from $\mathcal{T}_g$ are abstractly isomorphic; their arithmeticity depends therefore on how they are "positioned" in $PSL(2,\mathbb{R})$.

In general there is a criterion for arithmeticity due to Margulis and whose general strategy of proof follows the arithmeticity theorem. It is based on the commensurator of a lattice.

<u>**Def.**</u> 1.26 The commensurator $\mathrm{Comm}_G(\Gamma)$ of a subgroup Γ of a group G is the

set

$$\mathrm{Comm}_G(\Gamma) := \{ g \in G : \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable} \}.$$

Since commensurability is an equivalence relation it is easily seen that $\mathrm{Comm}_G(\Gamma)$ is a subgroup of G containing Γ.

Example - Exercise: $\mathrm{Comm}_{SL(n,\mathbb{R})}(SL(n,\mathbb{Z})) \supset SL(n,\mathbb{Q})$.

Given a connected simple Lie group G we have the following dichotomy:

Thm 1.27 Assume G is connected, simple, non-compact and let $\Gamma < G$ be a lattice. Then either $\mathrm{Comm}_G(\Gamma)$ is dense in G or it is discrete.
In the second case Γ is of finite index in $\mathrm{Comm}_G(\Gamma)$.

The arithmeticity criterion is then:

<u>Thm 1.28.</u> (Margulis) Let $\Gamma < G$ be a lattice, and G is connected, simple non-compact. Then Γ is arithmetic $\Longleftrightarrow$ $\mathrm{Comm}_G(\Gamma)$ is dense in G.

Now the proof of the arithmeticity theorem is relatively simple consequence of a result known as superrigidity. This result gives very strong information ~~abo~~, under the hypothesis of Thm A, on linear representations

$$\pi : \Gamma \longrightarrow GL(V)$$

where V is a finite dimensional vector space over $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$. In the sequel such a field will be called a local field of characteristic zero. ~~To state the superrigidity~~ Such a field k is locally compact and so are the groups $GL(V)$ where V is a finite dimensional k-vector space. We'll need to important notions:

<u>Def. 1.29</u> A representation
$$\pi : \Gamma \longrightarrow GL(V) \quad \text{is}$$
strongly irreducible if its restriction to any finite index subgroup $\Gamma' < \Gamma$ is still irreducible.

<u>Def 1.30</u>. An action by Γ on a compact metric space (X, d) is proximal if for every $(x, y) \in X^2$ there exists a sequence $(\gamma_n)_{n \geq 1}$ in Γ with $\lim_{n\to\infty} d(\gamma_n x, \gamma_n y) = 0$.

Given a finite dimensional k-vector space V, where k is a local field of characteristic zero, the projective space $\mathbb{P}(V) := k^\times \backslash V$ with its quotient topology is a compact metric space. A representation $\pi : \Gamma \longrightarrow GL(V)$ gives thus rise to a Γ-action on $\mathbb{P}(V)$ by homeomorphisms.

<u>Theorem B</u> (Margulis superrigidity)
Assume $\Gamma < G$ is a lattice in a connected simple Lie group G with $\operatorname{rank}(G) \geq 2$. Let $\pi : \Gamma \longrightarrow GL(V)$ be a representation where V is a finite dimensional vector space over a local field of characteristic zero. Assume:

(1) π is strongly irreducible.

(2) the Γ-action on $\mathbb{P}(V)$ is strongly proximal.

Then π extends continuously to a homomorphism $\pi_{ext} : G \to PGL(V)$.

Remarks 1.31.

(1) This theorem is called superrigidity because it implies Mostow's rigidity theorem when $\mathrm{rank}(G) \geq 2$.

(2) Let me give a very simplified explanation of why Superrigidity implies arithmeticity. For this, assume $G = SL(n, \mathbb{R})$, $n \geq 3$. For any $\sigma \in \mathrm{Aut}\,\mathbb{C}$ consider the homomorphism

$\pi_\sigma : \Gamma \longrightarrow SL(n, \mathbb{C})$ obtained by applying σ to all matrix coefficients of elements in Γ. The case $k = \mathbb{R}, \mathbb{C}$ of ThB implies the existence of $g \in SL(n, \mathbb{R})$ and a finite extension $\mathbb{K}/\mathbb{Q}$ such that $g \Gamma g^{-1} \subset SL(n, \mathbb{K})$. Assume $g = id$, $\mathbb{K} = \mathbb{Q}$. For every prime number p consider the hom.

$$\pi_p : \Gamma \longrightarrow SL(n, \mathbb{Q}_p)$$

obtained by composing the inclusion $\Gamma \hookrightarrow SL(n, \mathbb{Q})$ with $SL(n, \mathbb{Q}) \hookrightarrow SL(n, \mathbb{Q}_p)$. Since there are no non-trivial continuous homomorphisms $SL(n, \mathbb{R}) \longrightarrow SL(n, \mathbb{Q}_p)$ the case $k = \mathbb{Q}_p$ of the superrigidity theorem will imply that (oversimplyfying)

$$\pi_p(\Gamma) \subset SL(n, \mathbb{Z}_p).$$

Thus for every $\gamma \in \Gamma$, all matrix entries are in $\mathbb{Q}$ and in the p-adic integers $\forall p$. But this implies
$$\Gamma \subset SL(n,\mathbb{Z}).$$

While we will formulate the main steps leading to Thm B in their natural degree of generality, we will give a complete proof in the case $G = SL(n,\mathbb{R})$. Indeed no new ideas are required in order to treat the general case.

The rough plan of the course is the following:

Chapter 2: Invariant measures on homogeneous spaces.

Chapter 3: Examples of arithmetic lattices.

Chapter 4: Ergodic actions: the Howe-Moore theorem.

Chapter 5: Lattices and boundary theory.

Chapter 6: Proof of the Superrigidity Theorem.

Chapter 2. Invariant measures on homogeneous spaces.

Let X be a locally compact (Hausdorff) space. ~~Recall that~~ A linear functional

$$I : C_{00}(X) \longrightarrow \mathbb{R}$$

on the space of $\mathbb{R}$-valued ~~fo~~ continuous functions with compact support is positive if $I(f) \geq 0$ whenever $f \geq 0$. Recall that by the Riesz representation theorem such a functional is represented as

$$I(f) = \int_X f(x)\, d\mu(x)\,, \quad f \in C_{00}(X)$$

where μ is a positive Radon measure on X, that is a regular Borel measure that is finite on compact sets and positive. [See Rudin: Real and Complex Anal. Chapt. 2.]

<u>Thm 2.1.</u> (Haar) On a locally compact group G there is, up to positive scaling, a unique left invariant, non-zero, positive Radon measure.

For a short proof see A. Weil, "l'intégration dans les groupes topologiques et ses applications".

<u>Exercise 2.2.</u> Show the existence part of Thm 2.1 under the assumption that G is a Lie group.

The uniqueness statement is probably as important as the existence. This is illustrated in the following Corollaries.

<u>Corollary 2.3</u> Let dg be a left invariant Haar measure on G. There exists a continuous homomorphism $\Delta_G : G \longrightarrow \mathbb{R}^{\times}_{>0}$

with $$\int_G f(gh^{-1})\,dg = \Delta_G(h) \int_G f(g)\,dg \quad (*)$$
$\forall f \in C_{00}(G)$.

Proof: Define $J(f) := \int_G f(gh^{-1})\,dg$

and observe that J is left invariant.

Def. 2.4. $\Delta_G : G \longrightarrow \mathbb{R}^{\times}_{>0}$ is called the modular function and G is called unimodular if $\Delta_G \equiv 1$, equivalently any left invariant Haar measure is right invariant.

Corollary 2.5 Let dg be a left invariant Haar measure. Then
$$\int_G f(g^{-1})\,\Delta_G(g^{-1})\,dg = \int_G f(g)\,dg$$
$\forall f \in C_{00}(G)$.

Proof: Using $(*)$ one verifies that
$$J(f) := \int_G f(g^{-1})\,\Delta_G(g^{-1})\,dg \quad \text{defines}$$

a left invariant functional. Hence $\exists\, c > 0$ with

$$\int_G f(g^{-1})\,\Delta_G(g^{-1})\,dg = c \cdot \int_G f(g)\,dg$$

$\forall f \in C_{0,c}(G)$.

Actually since dg is a Radon measure this equality holds as well for $f = \chi_W$ whenever W is any open with compact closure: both sides are then finite. Let then $V \ni e$ be any open neighborhood with compact closure and $W = V \cap V^{-1}$. For $f = \chi_W$ we get:

$$\int_W \Delta_G(g^{-1})\,dg = c \cdot \int_W 1\,dg$$

or:
$$c = \frac{\int_W \Delta_G(g^{-1})\,dg}{\int_W dg}.$$

Using that Δ_G is continuous with $\Delta_G(e) = 1$ and shrinking W we get $c = 1$.

$\boxed{\Xi}$