Examples 2.6.

(1) $\mathbb{R}^n$: product of Lebesgue measure on $\mathbb{R}$: $dx_1 \cdots dx_n$.

(2) Compact groups as well as abelian groups are unimodular.

(3) $N = \left\{ \begin{pmatrix} 1 & \ddots & x_{ij} \\ 0 & \ddots & 1 \end{pmatrix} : x_{ij} \in \mathbb{R} \atop i<j \right\}$

Then $dn := \prod_{i<j} dx_{ij}$ is a left invariant Haar measure on N. This relies on: for $A, x \in N$,

$$(Ax)_{ij} = x_{ij} + \text{things not involving } x_{ij}.$$

Same argument shows dn is right invariant.

(4) $\mathbb{R}^\times_{>0}$: if dx is the Lebesgue measure on $\mathbb{R}$, $\frac{dx}{x}$ is a left Haar

measure on $\mathbb{R}^{\times}_{>0}$.

(5) $B^{+} = \left\{ \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} : \begin{array}{l} a_i > 0 \\ \prod_{i=1}^{n} a_i = 1 \end{array} \right\}$

$A^{+} = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} : \begin{array}{l} a_i > 0 \\ \prod_{i=1}^{n} a_i = 1 \end{array} \right\}$

Let da^{+} resp. dn be left Haar measures on A^{+} resp. N. Then

$$db^{+} = da^{+} dn$$

is a left Haar measure on B^{+}:

$$\int_{A^{+}} da^{+} \int_{N} dn\, f(a, \underbrace{n, a n}_{a, a(a^{-1}n, a) n})$$

So $\int_{N} dn\, f(a, n, an) = \int_{N} f(a, a n)$, since

$a^{-1}n, a \in N$.

Thus the whole $= \int_{A^{+}} da^{+} \int_{N} dn\, f(a, an)$

$$= \int_{A^+} da^+ \int_N dn\, f(an).$$

However B^+ is not unimodular, in fact

$$\Delta_{B^+}(an) = \prod_{i<j} \frac{a_i}{a_j}.$$

(6) Counting measure on a discrete group.

Now we turn to homogeneous spaces.
Let G be a l.c. group, $H < G$ a closed subgroup and $\pi : G \to G/H$ the canonical projection. Endowed with quotient topology G/H is locally compact, π is continuous and open, and:

$$G \times G/H \longrightarrow G/H$$
$$(g, xH) \longmapsto gxH$$

is a continuous action.

Let $L_g : G/H \longrightarrow G/H$ be the
$xH \longmapsto gxH$
homeo. of multiplication by g on the left.

<u>Def. 2.7.</u> A positive Radon measure μ on G/H is semi-invariant if

$$(L_g)_* \mu \quad \text{and} \quad \mu$$

are proportional $\forall g \in G$.

The proportionality factor $\chi(g) > 0$ gives then rise to a continuous homomorphism $\chi : G \longrightarrow \mathbb{R}^{\times}_{>0}$, called the modulus of the semi-invariant measure.

<u>Thm 2.8.</u> (A. Weil) Let $\chi : G \longrightarrow \mathbb{R}^{\times}_{>0}$ be a continuous homomorphism. There exists on G/H a semi invariant Radon measure with modulus χ iff

$$\chi\big|_H = \frac{\Delta_G|_H}{\Delta_H}.$$

This measure is unique up to scaling.
Given left Haar measures dg on G and dh on H there is a unique semi-invariant measure μ on G/H such that

$$\int_{G/H} d\mu(x) \int_H f(xh)\,dh = \int_G f(g)\,\chi(g)^{-1}\,dg$$

$$\forall f \in C_c(G).$$

<u>Remark 2.10</u> : The theorem implies that G/H admits a semi-invariant measure wrt some modulus iff $\Delta_H : H \longrightarrow \mathbb{R}^\times_{>0}$ extends to a continuous homomorphism $G \longrightarrow \mathbb{R}^\times_{>0}$. This is for instance the case if H is unimodular.

Before proving the theorem let us draw some easy consequences:

Corollary 2.11. (1) If H is unimodular there is a semi-invariant measure on G/H of modulus Δ_G.

(2) If H is unimodular and G/H is compact, the semi-invariant measure on G/H is invariant and G is unimodular.

(3) If $\Gamma < G$ is a discrete subgroup with G/Γ compact then Γ is a lattice in G.

Proof: (1) Clear from Remark 2.10.

(2) Since G/H is compact, $\mu(G/H) < +\infty$ and $\mu(L_g^{-1}(G/H)) = \mu(G/H)$. On the other hand

$$\mu(L_g^{-1}(G/H)) = \Delta_G(g)\,\mu(G/H)$$

$\forall g \in G$ and since $\mu(G/H) > 0$

this implies $\Delta_G(g) = 1 \ \forall g \in G$.

(3) Follows from (2).

$\boxed{\xi}$

As an application of Thm 2.8 we shall derive a formula for the Haar measure on $SL(n,\mathbb{R})$. To this end recall

$$B^+ = \left\{ \begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} : \begin{array}{c} a_i > 0 \\ \prod a_i = 1 \end{array} \right\}$$

and that by the Gram-Schmidt orthogonalization procedure every $g \in SL(n,\mathbb{R})$ has a unique decomposition:

$$g = b \cdot k \quad , b \in B^+, k \in SO(n).$$

As a result the product map

$$B^+ \times SO(n) \longrightarrow SL(n,\mathbb{R})$$

is a homeomorphism and thus

$$B^+ \longrightarrow SL(n,\mathbb{R})/SO(n)$$

is a B^+-equivariant homeo. Next observe that $SO(n)$ and $SL(n,\mathbb{R})$ are unimodular and hence $SL(n,\mathbb{R})/SO(n)$ carries an $SL(n,\mathbb{R})$-invariant measure which by uniqueness of Haar measure must be the image of a left invariant measure db^+ on B^+. Thus:

<u>Corollary 2.12</u> If dk is a Haar measure on $SO(n)$ and db^+ a left Haar measure on B^+ then

$$f \longmapsto \int_{B^+} db^+ \int_{SO(n)} f(b^+k)\, dk$$

defines a left Haar measure on $SL(n,\mathbb{R})$.

Finally we mention the following corollary that is a consequence of ~~the uniqueness~~ Thm 2.8:

<u>Corollary 2.13</u> Let $H_1 < H_2 < G$ be closed subgroups of a l.c. group G and assume H_1, H_2, G unimodular. Then there is a choice of invariant measures μ on G/H_1, ν on G/H_2 and α on H_2/H_1 s.t. $\forall f \in C_{00}(G/H_1)$

$$\int_{G/H_1} f(y)\,d\mu(y) = \int_{G/H_2} d\nu(x) \int_{H_2/H_1} f(x\xi)\,d\alpha(\xi).$$

The rest of the section is devoted to the proof of Thm 2.8. As a important tool we introduce an "averaging over H" operator, defined as follows: let dh be a left invariant Haar measure on H.

Given $f \in C_{00}(G)$, the function

$$\begin{aligned} H &\longrightarrow \mathbb{R} \\ h &\longmapsto f(gh) \end{aligned}$$

is in $C_{00}(H)$ for every $g \in G$ and thus we can define:

$$T_H f(g) := \int_H f(gh)\,dh .$$

The function $T_H f$ is then H-invariant on the right and we consider it as a function $G/H \longrightarrow \mathbb{R}$.

<u>Lemma 2.14</u> For every $f \in C_{00}(G)$, $T_H f \in C_{00}(G/H)$ and the map

$$T_H : C_{00}(G) \longrightarrow C_{00}(G/H)$$

is surjective. Moreover if $F \in C_{00}(G/H)$ is ≥ 0 we can choose $f \in C_{00}(G)$ with $T_H f = F$ and $f \geq 0$.

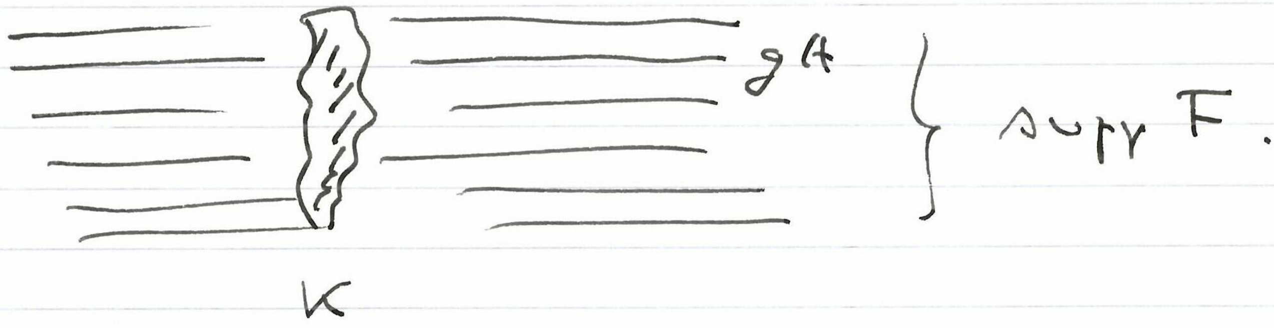
<u>Proof</u>: Let $\pi : G \longrightarrow G/H$ be the canonical projection.

(1) We leave the fact that $T_H f \in C_{0\circ}(G/H)$ as an exercise. To show continuity of $T_H f$ the essential point is to show that $f \in C_{00}(G)$ is uniformly continuous in the following sense: $\forall \varepsilon > 0$ $\exists U \ni e$ open such that $|f(x) - f(y)| < \varepsilon$ $\forall x, y$ such that $xy^{-1} \in U$.

(2) For every compact $C \subset G/H$ there is $K \subset G$ compact with $\pi(K) = C$. Indeed: $\forall x \in C$ pick $\tilde{x} \in G$ with $\pi(\tilde{x}) = x$ and $V_{\tilde{x}} \ni \tilde{x}$ open with compact closure. Then there is $x_1, \dots, x_n$ with $C \subset \bigcup_{i=1}^{n} \pi(V_{\tilde{x}_i})$ (remember that π is open!); clearly $\bigcup_{i=1}^{n} \overline{V_{\tilde{x}_i}} \subset G$ is compact, $\pi\left(\bigcup_{i=1}^{n} \overline{V_{\tilde{x}_i}}\right) \supset C$ and

hence $K := \pi^{-1}(C) \cap \left(\bigcup_{i=1}^{n} \overline{V}_{\tilde{x}_i} \right)$ does the job.

(3) Let $F \in C_{00}(G/H)$ and $K \subset G$ compact with $\pi(K) = \operatorname{supp} F$.



Choose $l \in C_{00}(G)$ with $l \geq 0$ and $l > 0$ on K. Define

$$f(g) = \begin{cases} \dfrac{l(g)\, F(gH)}{\int_H l(gh)\, dh} & \text{if } \int_H l(gh)\, dh > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then f is with compact support since

$$\operatorname{supp} f \subset \operatorname{supp} l.$$

Let $\Omega_1 = \{ g \in G : l(g) > 0 \} \cdot H$

Then Ω_1 is open; for every $g \in \Omega_1$ $\{h : \ell(gh) > 0\}$ is an open, non-empty subset of H hence of positive dh-measure. This implies that

$$g \longmapsto \frac{1}{\int_H \ell(gh)\,dh}$$

is continuous on Ω_1 and so is f.

Now $\Omega_1 \supset \pi^{-1}(\operatorname{supp} F)$ by the definition of ℓ; $\Omega_2 := [\pi^{-1}(\operatorname{supp} F)]^c$ is then open and $f|_{\Omega_2} = 0$. Thus since $G = \Omega_1 \cup \Omega_2$ we conclude that f is continuous. $\boxed{\wr\wr}$

<u>Proof of Thm 2.8</u>:

Assume that $\mathcal{X}$ satisfies the assumptions of the Theorem. We want to define

$\exists$ a positive linear functional on $C_{00}(G/H)$ in the following way: for $F \in C_{00}(G/H)$ pick (lemma 2.14) $f \in C_{00}(G)$ with $T_H f = F$ and define

$$J(F) := \int_G f(g)\chi(g^{-1})\,dg.$$

The main point is to show that this is well defined; this will follow if we show that $F = 0$ implies $\int_G f(g)\chi(g^{-1})\,dg$ whenever $T_H f = F$.

Since $\pi(\operatorname{supp} f) \subset G/H$ is compact we can find $l \in C_{00}(G)$ with $T_H l = 1$ on $\pi(\operatorname{supp} f)$. Then:

$$\int_G f(g)\chi(g^{-1})\,dg = \int_G f(g)\, T_H l(g)\, \chi(g^{-1})\,dg$$

$$= \int_G f(g)\chi(g^{-1}) \int_H l(gh)\,dh\;dg$$

$$\int_H dh \underbrace{\int_G f(g)\,\chi(g^{-1})\,\ell(gh)\,dg}_{\Delta_G(h^{-1})\int_G f(gh^{-1})\,\ell(g)\,\chi(g^{-1}h)\,dg}$$

$$= \int_G dg\, \ell(g)\,\chi(g^{-1}) \underbrace{\int_H dh\, f(gh^{-1})\,\Delta_G(h^{-1})\,\chi(h)}_{\int_H dh\, f(gh)\,\underbrace{\Delta_G(h)\,\chi(h^{-1})\,\Delta_H(h^{-1})}_{=1}}$$

$$= \int_G dg\, \ell(g)\,\chi(g^{-1}) \underbrace{T_H f(g)}_{F(g)}$$

$$= 0.$$

Thus $J(F) = \int_G f(g)\,\chi(g^{-1})\,dg$

is a positive functional, hence there is a unique positive Radon measure μ

on G/H with $I(F) = \int_{G/H} F(\dot{g})\, d\mu(\dot{g})$

which implies:

$$\int_{G/H} d\mu(\dot{g}) \int_H f(gh)\, dh = \int_G f(g)\, \chi(g^{-1})\, dg.$$

Replacing f by $({}_x f)(g) = f(xg)$

We got:

$$\int_{G/H} d\mu(\dot{g}) \int_H f(xgh)\, dh = \int_G f(xg)\, \chi(g^{-1})\, dg$$

$$= \chi(x) \int_G f(g)\, \chi(g^{-1})\, dg$$

and Thus

$$\int_{G/H} d\mu(\dot{g})\, F(x\dot{g}) = \chi(x) \int_{G/H} F(\dot{g})\, d\mu(\dot{g})$$

Which shows the semi invariance. $\blacksquare$

Uniqueness is obtained in the following way:

assume μ is a positive semi-invariant Radon measure on G/H with modulus

$$\chi : G \longrightarrow \mathbb{R}^{\times}_{>0}.$$

Define for $f \in C_{00}(G)$:

$$I(f) := \int_{G/H} d\mu(g)\, T_H(f \cdot \chi)(g)$$

Then $\forall x \in G$:

$$T_H({}_x f \cdot \chi)(g) = T_H({}_x f_x \chi)(g) \frac{1}{\chi(x)}$$

$$= \frac{1}{\chi(x)} T_H(f \cdot \chi)(xg)$$. And thus

by quasiinvariance of μ:

$$I({}_x f) = \int_{G/H} d\mu(g)\, T_H(f \cdot \chi)(xg) \frac{1}{\chi(x)}$$

$$= \int_{G/H} d\mu(g)\, T_H(f \cdot \chi)(g)$$

Which implies that $I(f) = \int_G f(g)\, dg$

where dg is a left invariant Haar measure. Thus:

$$\int_{G/H} d\mu(g) \int_H f(gh)\,dh = \int_G f(g)\,\chi(g)^{-1}\,dg$$

Since $T_H : C_{00}(G) \longrightarrow C_{00}(G/H)$ is surjective this determines μ. $\boxed{\xi}$

With a little bit more work one can extend Corollary 2.13 to L^1-functions and obtains

<u>Corollary 2.15</u> Under the assumption of Corollary 2.13 and with the choices of μ, ν and α we have:
For $f \in L^1(G/H_1)$, we have that for almost every $x \in G$, $z \longrightarrow f(xz)$ is in $L^1(H_2/H_1)$, the function

$$G/H_2 \longrightarrow [0, \infty]$$

$$x \longmapsto \int_{H_2/H_1} |f(x\xi)|\, d\alpha(\xi)$$

is in $L^1(G/H_2)$ and the formula

$$\int_{G/H_1} f(y)\, d\mu(y) = \int_{G/H_2} d\nu(x) \int_{H_2/H_1} f(x\xi)\, d\alpha(\xi)$$

holds.

We will refrain from giving the details and just mention that it follows from 2.13 that the operator:

$$T : C_{00}(G/H_1) \longrightarrow C_{00}(G/H_2)$$

$$Tf(x) := \int_{H_2/H_1} f(x\xi)\, d\alpha(\xi)$$

is bounded in L^1, in fact:

$$\| Tf \|_{L^1(G/H_2)} \leq \| f \|_{L^1(G/H_1)}.$$

so that T extends to a bounded operator $L^1(G/H_1) \to L^1(G/H_2)$.

On gets stronger results with the additional assumption that G is second countable. In this case G as a Borel space is isomorphic to $G/H \times H$, with product Borel structure and all measures involved are σ-finite. This allows to apply Fubini's theorem and we get:

<u>Theorem 2.11.</u> Let G be a second countable l.c. group and $H_1 < H_2 < G$ closed subgroups s.t. H_1, H_2 and G are unimodular. Let μ, ν and α be compatible choices of invariant measures on G/H_1, G/H_2 and H_2/H_1.

(1) $f \in L^1(G/H_1)$ iff for a.e. $g \in G$, $z \longrightarrow f(gz)$ is in $L^1(H_2/H_1)$, the function

$$G/H_2 \longrightarrow [0,\infty)$$
$$g \longmapsto \int_{H_2/H_1} |f(gz)|\, d\alpha(z)$$

is in $L^1(G/H_2)$ and then the formula in Cor. 2.13 holds.

(2) Assume $f : G/H_1 \longrightarrow [0,\infty]$ is measurable. Then the formula in Cor. 2.13 holds.

<u>Example</u>: ~~Sou~~ The assumptions in Thm 2.16 are important. Indeed let $G = \mathbb{R} \times \mathbb{R}_d$ where $\mathbb{R}_d$ is endowed with discrete topology and $f : \mathbb{R} \times \mathbb{R}_d \longrightarrow [0,1]$,

$f(x,y) = 1$ if $x = y$, $0 \leq x \leq 1$ and zero otherwise.

Then with $H_1 = \mathbb{R}$ and $H_2 = \mathbb{R}_d$

one verifies that

$$\int_{G/H_1} d\mu_c(g) \int_{H_1} d\cancel{\mathcal{L}}(h)\, f(g+h)$$

$$\neq \int_{G/H_2} d\cancel{\mathcal{L}}(g) \int_{H_2} d\mu_c(h)\, f(g+h).$$