After having established a criterion saying when G/H carries a semi-invariant measure we are still faced in general with the problem of the existence of a measure on G/H that is invariant in some reasonable sense. We begin with two definitions stated in the degree of generality needed in the chapter on ergodic actions.

Let $R \times X \to X$ be a continuous action by a locally compact group R on a locally compact space X.

<u>Def. 2.18</u>: Two positive Radon measures μ_1, μ_2 on X are equivalent if they share the same family of null sets.

A measure class is then an equivalence class of measures.

<u>Def. 2.19:</u> We say that a positive Radon measure μ is quasi-invariant for the R-action if R preserves the measure class of μ. In other words

$$\mu(E) = 0 \iff \mu(gE) = 0 \quad \forall g \in R.$$

If now G is second countable and μ_G a left Haar measure then there exists $\psi : G \to [0, \infty)$ continuous with $\psi(g) > 0$ $\forall g \in G$ and $\int_G \psi \, d\mu_G < +\infty$.

Indeed G can be written as a countable union $\bigcup_{n=1}^{\infty} K_n$ of compact subsets. Let $\psi_n \in C_{00}(G)$ with $\psi_n \geq 0$ and $\psi_n > 0$ on K and define

$$\psi = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\psi_n}{\int \psi_n \, d\mu_G}.$$

Let $\pi : G \to G/H$ be the canonical projection and define $\mu := \pi_*(\psi \cdot \mu_G)$.

Then $\forall E \subset G/H$ we have $\forall g \in G$:

$$\mu(gE) = \int_{g \cdot \pi^{-1}(E)} \psi(x) \, d\mu_G(x)$$

By construction the measures μ_G and $\psi \mu_G$ are equivalent and hence

$$\mu(gE) = 0 \iff \mu_G(g\pi^{-1}(E)) = 0.$$

$$\iff \mu_G(\pi^{-1}(E)) = 0$$

$$\mu(E) = 0 \iff$$

which shows that μ is quasiinvariant.

In fact

Thm 2.20. Assume G is l.c. s.c. and $H < G$ is a closed subgroup. There is a unique G-invariant measure class on G/H. If μ is any representative thereof $\mu(E) = 0 \iff \mu_G(\pi^{-1}(E)) = 0$.

This is based on the following

Lemma 2.21 Under the same assumptions there exists a Borel subset $B \subset G$ such that

(1) $gH \cap B$ consists of 1 point $\forall g \in G$

(2) $\forall K \subset G$ compact, $KH \cap B$ has compact closure in G.

From this one deduces easily that the map $s : G/_H \longrightarrow B$

$$gH \mapsto gH \cap B$$

is a Borel isomorphism sending compact sets to sets with compact closure.

For details concerning Thm 2.20 and lemma 2.21 we refer to G. W. Mackey "Induced representations of locally compact groups I", Annals of Math. 1952, pp 101–139

see § 1.

Chapter 3 Examples of arithmetic lattices.

First we start ~~by~~ with the "fundamental theorem of geometry of numbers" due to Minkowski.

On $\mathbb{R}^n$ we fix the Lebesgue measure $\mathcal{L}$ normalized so that

$$\mathcal{L}([0,1]^n) = 1.$$

$\mathcal{L}$ is a Haar measure on $\mathbb{R}^n$ and given any lattice $\Lambda < \mathbb{R}^n$ it follows from Thm 2.8 (and its improvement 2.15) that there is a unique $\mathbb{R}^n$-invariant measure μ_Λ on $\mathbb{R}^n/\Lambda$ such that $\forall f \in L^1(\mathbb{R}^n)$:

$$(*) \quad \int_{\mathbb{R}^n/\Lambda} d\mu_\Lambda(\dot{x}) \sum_{\lambda \in \Lambda} f(x+\lambda) = \int_{\mathbb{R}^n} d\mathcal{L}(y) f(y).$$

where all integrals involved are absolutely convergent. To simplify notation we will denote by $\mathrm{Vol}(E)$ the measure of any measurable subset $E \subset \mathbb{R}^n$, wrt $\mathcal{L}$, or $E \subset \mathbb{R}^n/\Lambda$ wrt μ_Λ.

Recall that a subset $V \subset \mathbb{R}^n$ is called balanced if $V = -V$.

Theorem 3.1. Let $\Lambda < \mathbb{R}^n$ be a lattice and $V \subset \mathbb{R}^n$ a compact convex balanced subset with

$$\mathrm{Vol}(V) \geq 2^n \, \mathrm{Vol}(\mathbb{R}^n/\Lambda).$$

Then $V \cap (\Lambda \setminus \{0\}) \neq \emptyset$.

Proof: Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\Lambda$ the canonical projection.

Claim: π is not injective on $\frac{1}{2}V$.

Let's show Claim $\Rightarrow$ Thm. There are $x \neq y$ in $\frac{1}{2}V$ with $\pi(x) = \pi(y)$, that is $x - y =: \lambda \in \Lambda$ and $\lambda \neq 0$.

But: $x \in \frac{1}{2}V$, $y \in \frac{1}{2}V$ hence $-y \in \frac{1}{2}V$ since V is balanced. Hence

$0 \neq \lambda = x - y \in \frac{1}{2}V + \frac{1}{2}V \subset V$ since V is convex.

We prove the claim by contradiction. Assume that π is injective on $\frac{1}{2}V$.

First we observe that this implies $\pi(\frac{1}{2}V) \neq \mathbb{R}^n/_\Lambda$.

There are many ways to prove this; for instance otherwise $\frac{1}{2}V$ would be homeomorphic to $\mathbb{R}^n/_\Lambda$, but $\pi_1(\frac{1}{2}V) = (0)$ while $\pi_1(\mathbb{R}^n/_\Lambda) \simeq \Lambda$. Here is an elementary one. Otherwise, if $\pi(\frac{1}{2}V) = \mathbb{R}^n/_\Lambda$,

$$\mathbb{R}^n = \bigcup_{\lambda \in \Lambda} (\lambda + \tfrac{1}{2}V).$$

This union is dijoint because π is injective on $\frac{1}{2}V$; furthermore since Λ is discrete and $\frac{1}{2}V$ compact, this union is locally finite. Thus $\mathbb{R}^n$ would be the union of the disjoint ~~closed~~ sets $\frac{1}{2}V$ and $\bigcup_{\lambda \neq 0} (\lambda + \frac{1}{2}V)$, a contradiction.

Since π is injective on $\frac{1}{2}V$ it follows from $(*)$ that

$$\mathrm{Vol}\left(\pi\left(\tfrac{1}{2}V\right)\right) = \mathrm{Vol}\left(\tfrac{1}{2}V\right) \geq \mathrm{Vol}\left(\mathbb{R}^n/\Lambda\right).$$

Hence

$$\mathrm{Vol}\left(\pi\left(\tfrac{1}{2}V\right)\right) = \mathrm{Vol}\left(\mathbb{R}^n/\Lambda\right).$$

Now $\pi\left(\frac{1}{2}V\right) \subset \mathbb{R}^n/\Lambda$ being compact is closed and ~~$\mathrm{Vol}(\mathbb{R}^n/\Lambda)$~~ μ_Λ being $\mathbb{R}^n$-invariant is strictly positive on non-empty open sets. Hence

$$\pi\left(\tfrac{1}{2}V\right) = \mathbb{R}^n/\Lambda,$$ a contradiction. $\boxed{S}$

3.1. $SL(n,\mathbb{Z})$ is a lattice in $SL(n,\mathbb{R})$.

In order to proceed, recall that $SL(n,\mathbb{R})$ acts transitively on the set

$$\mathcal{R}^{(1)} = \{ \Lambda < \mathbb{R}^n : \Lambda \text{ is a lattice with } \operatorname{vol}(\Lambda \backslash \mathbb{R}^n) = 1 \}$$

and the stabiliser of $\mathbb{Z}^n$ is $SL(n,\mathbb{Z})$.

Via the equivariant bijection

$$SL(n,\mathbb{R})/_{SL(n,\mathbb{Z})} \longrightarrow \mathcal{R}^{(1)}$$

$$[g] \longmapsto g(\mathbb{Z}^n)$$

we transport the topology of $SL(n,\mathbb{R})/_{SL(n,\mathbb{Z})}$ to $\mathcal{R}^{(1)}$; in fact this topology can be characterized intrinsically as the Chabauty topology defined in the set of all closed subgroups of $\mathbb{R}^n$. More on this latter.

Now since both $SL(n,\mathbb{R})$ and $SL(n,\mathbb{Z})$ are unimodular, by Thm 2.8, there is an $SL(n,\mathbb{R})$ - invariant positive Radon measure on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$, and whose image on $\mathcal{R}^{(1)}$ we denote by μ.

Given $\Lambda < \mathbb{R}^n$ a lattice we call $\lambda \in \Lambda$ primitive if $\forall n \geq 2$, $\lambda/n \notin \Lambda$ and denote by Λ_{prim} the set of primitive vectors of Λ. Since Λ is discrete, every nonzero vector in Λ is an integer multiple of a primitive one. For example:

$$\mathbb{Z}^n_{prim} = \left\{ v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{Z}^n \setminus (0) \right.$$

such that $v_1, \dots, v_n$ are coprime $\Big\}$.

Given $f : \mathbb{R}^n \to \mathbb{R}$ let $\forall \Lambda \in \mathcal{R}^{(1)}$

$$F(\Lambda) = \sum_{\lambda \in \Lambda_{prim}}' f(\lambda)$$

whenever defined.

With this we have:

Theorem 3.2

Let $n \geq 2$ and assume $SL(n-1, \mathbb{Z})$ is a lattice in $SL(n-1, \mathbb{R})$. Then there is a constant $c_n > 0$ such that

$$\int_{\mathcal{R}^{(1)}} F(\Lambda)\, d\mu(\Lambda) = c_n \cdot \int_{\mathbb{R}^n} f(x)\, d\mathcal{L}(x)$$

whenever either $f : \mathbb{R}^n \longrightarrow [0, \infty]$ is measurable, in which case $F : \mathbb{R}^n \longrightarrow [0, \infty]$ is, or $f \in L^1(\mathbb{R}^n)$ in which case $F \in L^1(\mathcal{R}^{(1)}, \mu)$.

Corollary 3.3. $SL(n, \mathbb{Z})$ is a lattice in $SL(n, \mathbb{R})$, $\forall n \geq 1$.

Proof: By recurrence on n:

clearly $SL(1,\mathbb{Z}) = SL(1,\mathbb{R})$ and thus the assertion is true for $n=1$. Assume $n \geq 2$ and $SL(n-1,\mathbb{Z})$ is a lattice in $SL(n-1,\mathbb{R})$. Let $V = [-1,1]^n$ and apply the formula in Thm 3.2 to $f = \chi_V$.
Now V is compact, convex, balanced with $\mathrm{Vol}(V) = 2^n \geq 2^n \cdot 1 = 2^n \mathrm{Vol}(\mathbb{R}^n/\Lambda)$ $\forall \Lambda \in \mathcal{R}^{(1)}$.

As a result $V \cap \Lambda \setminus \{0\} \neq \emptyset$ $\forall \Lambda \in \mathcal{R}^{(1)}$.
But since V is convex and contains 0, if $\lambda \in V \cap \Lambda$, $\lambda \neq 0$ then $\lambda/n \in V$ $\forall n \geq 2$; now choose n so that $\lambda/n \in \Lambda_{prim}$ and this implies:

$$V \cap \Lambda_{prim} \neq \emptyset$$

and thus:

$$F(\Lambda) = \sum_{\lambda \in \Lambda_{prim}}' \chi_V(\lambda) \geq 1 \quad \forall \Lambda \in \mathcal{R}^{(1)}.$$

Thus:

$$\mu(\mathcal{R}^{(1)}) \leq \int_{\mathcal{R}^{(1)}} F(\Lambda)\,d\mu(\Lambda) = c_n \int_{\mathbb{R}^n} \chi_V(x)\,d\mathcal{L}(x)$$

$$= 2^n c_n < +\infty.$$

$\boxed{\xi}$

We precede the proof of Thm 3.2 with the following remark:

<u>Remark 3.4</u> Let G be l.c.s.c. unimodular $\Gamma < G$ discrete μ_G a left Haar measure on G and $\mu_{G/\Gamma}$ the compatible G-invariant measure on G/Γ. Let $\mathcal{F} \subset G$ be a Borel fundamental domain, that is, $\mathcal{F}$ is a Borel set, $\mathcal{F}\gamma \cap \mathcal{F} = \phi \;\; \forall \gamma \in \Gamma \setminus \{e\}$ and $\bigcup_{\gamma \in \Gamma} \mathcal{F}\gamma = G$. Applying Thm 2.11 with $H_1 = (e)$ and $H_2 = \Gamma$ to $f = \chi_{\mathcal{F}}$

we get

$$\mu_G(\mathcal{F}) = \int_G \chi_{\mathcal{F}}(g)\, d\mu_G(g)$$

$$= \int_{G/\Gamma} d\mu_{G/\Gamma}(x) \underbrace{\sum_{\gamma\in\Gamma} \chi_{\mathcal{F}}(x\gamma)}_{\equiv 1}$$

that is $\mu_G(\mathcal{F}) = \mu_{G/\Gamma}(G/\Gamma)$ and hence

Γ is a lattice iff $\mu_G(\mathcal{F}) < +\infty$.

~~Proof of Thm 3.2~~

~~Lemma 3.5~~.

Let $N = \mathrm{Stab}_{SL(n,\mathbb{R})}(e_1) = \left\{ \begin{pmatrix} 1 & v \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix} : \begin{array}{l} A \in SL(n-1,\mathbb{R}) \\ v \in \mathbb{R}^{n-1} \end{array} \right\}$.

and $N(\mathbb{Z}) := N \cap SL(n,\mathbb{Z})$.

Lemma 3.5 If $SL(n-1,\mathbb{Z})$ is a lattice in $SL(n-1,\mathbb{R})$ then $N(\mathbb{Z})$ is a lattice in N.

Proof. Identify N setwise with $\mathbb{R}^{n-1} \times SL(n-1,\mathbb{R})$. The group law takes the form:

$$(v_1, A_1)(v_2, A_2) = (v_2 + v_1 A_2, A_1 A_2)$$

so that the Haar measure μ_N on N is the product of $\mathcal{L}_{\mathbb{R}^{n-1}}$ and the Haar measure $\mu_{SL(n-1,\mathbb{R})}$. Now $(-1,1]^{n-1} := B$ is a Borel fundamental domain for $\mathbb{Z}^{n-1}$ in $\mathbb{R}^{n-1}$; if $\mathcal{F} \subset SL(n-1,\mathbb{R})$ is a Borel fundamental domain for $SL(n-1,\mathbb{Z})$ in $SL(n-1,\mathbb{R})$ we know that $\mu_N(B \times \mathcal{F})$

$$= \mathcal{L}_{\mathbb{R}^{n-1}}(B) \cdot \mu_{SL(n-1,\mathbb{R})}(\mathcal{F}) < +\infty.$$

Next: let $(v, A) \in N$: there is $A' \in SL(n-1, \mathbb{Z})$ with $AA' \in \mathcal{F}$, hence,

$$(v, A)(0, A') = (vA', AA') \in \mathbb{R}^{n-1} \times \mathcal{F}.$$

Now take $\gamma \in \mathbb{Z}^{n-1}$ with $vA' + \gamma \in \mathcal{B}$.

Then $(v, A)(0, A')(\gamma, Id) \in \mathcal{B} \times \mathcal{F}$.

$\boxed{\xi}$

<u>Proof of Thm 3.2</u>

Consider: $N(\mathbb{Z}) < N < SL(n, \mathbb{R})$

and let ν, α, ξ be compatible choices of invariant measures on resp. $SL(n,\mathbb{R})/N$, $N/N(\mathbb{Z})$ and $SL(n,\mathbb{R})/N(\mathbb{Z})$.

The orbit map

$$SL(n,\mathbb{R})/N \longrightarrow \mathbb{R}^n \setminus (0)$$

is a $SL(n,\mathbb{R})$-equiv. homeo; hence the Lebesgue measure $\mathcal{L}$ on $\mathbb{R}^n$

corresponds to a positive multiple of ν, which we may assume to be 1. Let $f: \mathbb{R}^n \to \mathbb{R}$ be as in Thm 3.2. Then

$$\int_{SL(n,\mathbb{R})/N} f(ge_1)\, d\nu(g) = \int_{\mathbb{R}^n} f(x)\, d\mathcal{L}(x).$$

Now we consider $g \longmapsto f(ge_1)$ as a function on $SL(n,\mathbb{R})/N(\mathbb{Z})$ and apply Thm 2.16 and obtain:

$$\int_{SL(n,\mathbb{R})/N(\mathbb{Z})} f(ge_1)\, d\rho(g) = \int_{SL(n,\mathbb{R})/N} d\nu(g) \int_{N/N(\mathbb{Z})} f(g\zeta e_1)\, d\alpha(\zeta) \quad (*)$$

By lemma 3.5, $N(\mathbb{Z})$ is a lattice in N and evidently $\zeta \longmapsto f(g\zeta e_1)$ is constant on $N/N(\mathbb{Z})$. As a result

$$(*) = \alpha\left(N/N(\mathbb{Z})\right) \int_{SL(n,\mathbb{R})/N} d\nu(g)\, f(ge_1)$$

$$= \alpha\left({}^{N}\!/_{N(\mathbb{Z})}\right) \int_{\mathbb{R}^n} f(x)\, d\mathcal{L}(x).$$

Now consider: $N(\mathbb{Z}) < SL(n,\mathbb{Z}) < SL(n,\mathbb{R})$.

There is a choice μ of $SL(n,\mathbb{R})$-inv. measure on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ such that Cor. 2.16 holds with counting measures on $SL(n,\mathbb{Z})/N(\mathbb{Z})$. Thus:

$$\int_{SL(n,\mathbb{R})/N(\mathbb{Z})} f(ge_1)\, d\xi(g) = \int_{SL(n,\mathbb{R})/SL(n,\mathbb{Z})} d\mu(g) \sum_{\gamma \in SL(n,\mathbb{Z})/N(\mathbb{Z})} f(g\gamma e_1).$$

We will conclude the proof of the theorem by interpreting the sum in the RHS.

In fact: $SL(n,\mathbb{Z})\cdot e_1 = (\mathbb{Z}^n)_{prim}$

and hence if $\Lambda = g(\mathbb{Z}^n)$ we conclude

$$\Lambda_{prim} = g(\mathbb{Z}^n_{prim}) = g(SL(n,\mathbb{Z})\cdot e_1).$$

But this implies precisely that

$$\sum_{\gamma \in SL(n,\mathbb{Z})/N(\mathbb{Z})} f(g\gamma e_1) = \sum_{\lambda \in \Lambda_{prim}} f(\lambda) = F(\Lambda).$$

$\boxtimes$

~~But then the above formula implies~~

$$\cancel{\mu\left(\mathcal{R}^{(1)}\right) \leq \int_{\mathcal{R}^{(1)}} F(\Lambda)\, d\mu(\Lambda) = C_n \cdot 2^n.}$$

Next we are going to adress the question of characterizing subsets of $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ with compact closure. In fact we will treat this problem in the model $\mathcal{R}^{(1)}$ of lattices in $\mathbb{R}^n$ of covolume 1. In fact $\mathcal{R}^{(1)}$ is a subset of $\mathcal{L}(\mathbb{R}^n)$ the set of closed subgroups of $\mathbb{R}^n$. This set can be endowed with a topology which makes it a compact metrisable space; this top. is called the Chabauty topology and can be defined on the set $\mathcal{S}(G)$ of closed subgroups of any locally compact group.

Let $B \subset \mathbb{R}^n$ be any open ball of finite radius center at 0. For closed subgroups H_1, H_2 in $\mathbb{R}^n$ set

$$d_B(H_1, H_2) = \sup_{\substack{\lambda_1 \in H_1 \cap B \\ \lambda_2' \in H_2 \cap B}} \left\{ d(\lambda_1, H_2 \cap B), d(\lambda_2', H_1 \cap B) \right\}$$

Then we say that $H_n \longrightarrow H$ if

$d_B(H_n, H) \longrightarrow 0$ for every such ball B. In fact

$$d(H, H') := \int_0^\infty d_{B(0,r)}(H, H')\, e^{-r}\, dr$$

defines a distance on $\mathcal{S}(\mathbb{R}^n)$.

Endow $\mathcal{R}^{(1)}$ with the induced topology.

Lemma 3.6 $SL(n,\mathbb{R})/SL(n,\mathbb{Z}) \longrightarrow \mathcal{R}^{(n)}$

$$[g] \longmapsto g(\mathbb{Z}^n)$$

is a homeomorphism.

Proof: left as an exercise. $\boxed{R}$

We have then

Thm. 3.7 (Mahler's compactness Cr.)

A subset $M \subset \mathcal{R}^{(n)}$ is relatively compact if and only if there exists $U \subset \mathbb{R}^n$ neighborhood of 0 s.t. $\Lambda \cap U = \{0\}$ $\forall \Lambda \in M$.

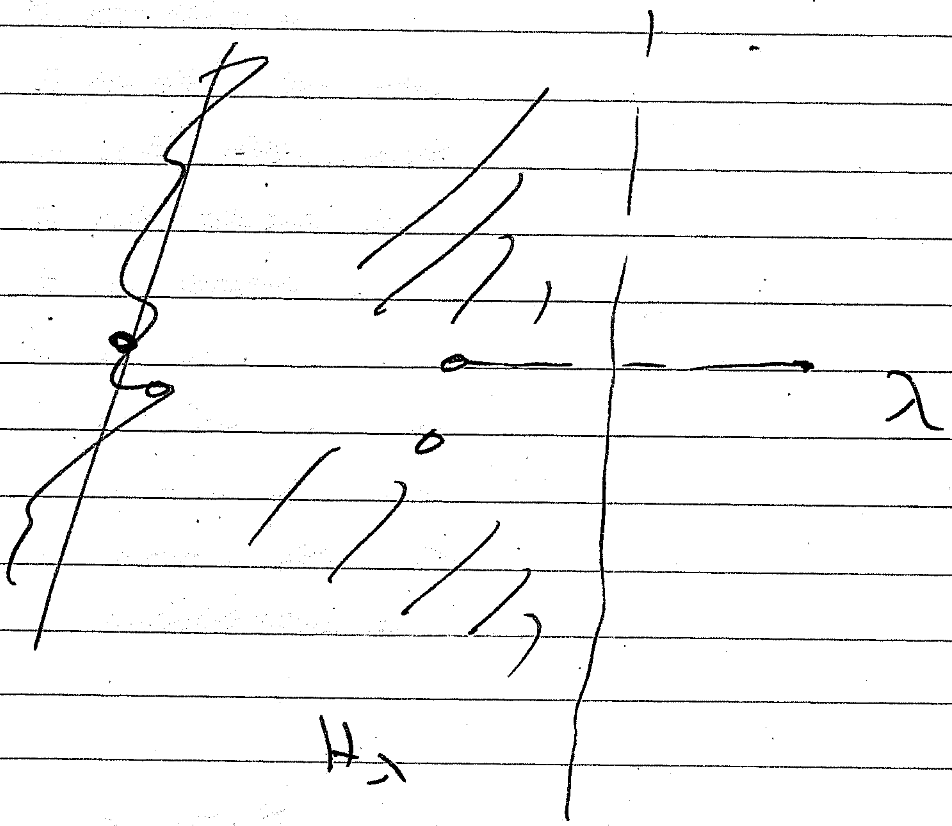
In other words, there is an $\varepsilon > 0$ s.t. every $\lambda \neq 0$ in $\lambda \in \Lambda$, satisfies $\|\lambda\| \geq \varepsilon$ for $\Lambda \in M$.

We will need the following which is of independent interest: Let $\Lambda < \mathbb{R}^n$ be a ~~lattice~~ discrete subgroup. Define

$$C_\Lambda = \{ x \in \mathbb{R}^n : d(x,0) \leq d(x,\lambda) \ \forall \lambda \in \Lambda \}$$

$$= \bigcap_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} H_\lambda$$

where $H_\lambda = \{ x \in \mathbb{R}^n : d(x,0) \leq d(x,\lambda) \}$



is a closed half space. Then C_Λ is closed convex.

<u>Lemma 3.8.</u> C_Λ is a fundamental domain for the Λ-action on $\mathbb{R}^n$, that is:

(1) $\pi(C_\Lambda) = \mathbb{R}^n / \Lambda$

(2) If $\pi(x) = \pi(y)$, $x \neq y \in C_\Lambda$ then $\{x, y\} \subset \partial C_\Lambda$.

<u>Proof:</u>

(1) Let $x \in \mathbb{R}^n$: since Λ is discrete, closed, $\{d(x, \mu) : \mu \in \Lambda\}$ has a minimum say at $\lambda \in \Lambda$. So

$$d(x, \lambda) \leq d(x, \mu) \quad \forall \mu \in \Lambda$$

and hence $d(x-\lambda, 0) \leq d(x-\lambda, \mu)$.

$\Rightarrow x - \lambda \in C_\Lambda$.

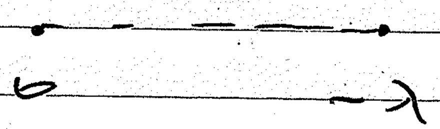
(2) Say $x \in C_\lambda$ and $x + \lambda \in C_{\Lambda}$. Then

$$d(x,0) \leq d(x,-\lambda) = d(x+\lambda, 0)$$

and $d(x+\lambda, 0) \leq d(X+\lambda, \lambda) = d(x,0)$.

Hence $\quad d(x,0) = d(x+\lambda, 0) = d(x,-\lambda)$

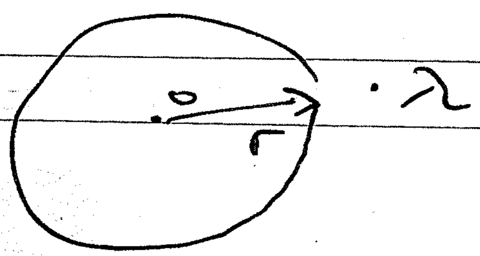
$$\cdot\, x \in \partial H_{-\lambda}$$



$\boxtimes$

<u>Lemma 3.9</u> For every $r > 0$ there is $c(r) > 0$ s.t. whenever $\Lambda \cap B(0,r) = \{0\}$ for $\Lambda \in \mathcal{R}^{(1)}$ then $S_\Lambda := \left(\Lambda \cap B(0, c(r))\right) \setminus 0$ generates Λ.

<u>Proof:</u> From $\Lambda \cap B(0,r) = \{0\}$ we deduce



that $C_\Lambda \supseteq B(0, \frac{r}{2})$. Now since

$Vol(C_\Lambda) = Vol(\Lambda \backslash \mathbb{R}^n) = 1$, we

deduce that $\forall x \in C_\Lambda$, the volume

of the convex hull of x and C_Λ is ≤ 1

Hence $\|x\| \leq c_1(r)$ and

$$C_\Lambda \subset B(0, c_1(r)).$$

Now $$C_\Lambda = \bigcap_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} H_\lambda = \bigcap_{\lambda \in \Lambda} \left(H_\lambda \cap B(0, c_1(r))\right)$$

$$= \bigcap_{0 < \|\lambda\| < 2c_1(r)} \left(H_\lambda \cap B(0, c_1(r))\right)$$

$$= \bigcap_{\lambda \in S_\lambda} H_\lambda \quad \text{with } c(r) = 2c_1(r).$$

Now let $\Lambda' < \Lambda$ be the subgroup

generated by S_Λ.

Clearly $C_{\Lambda'} = \bigcap_{\lambda \in X'} H_\lambda \subset \bigcap_{\lambda \in S_\Lambda} H_\lambda = C_\Lambda$

hence $\mathrm{Vol}(\Lambda' \backslash \mathbb{R}^n) = \mathrm{Vol}(C_{\Lambda'}) \leq \mathrm{Vol}(C_\Lambda)$

$= \mathrm{Vol}(\Lambda \backslash \mathbb{R}^n)$

On the other hand, $\Lambda' < \Lambda$, implies

$$\mathrm{Vol}(\Lambda' \backslash \mathbb{R}^n) = [\Lambda : \Lambda'] \, \mathrm{Vol}(\Lambda \backslash \mathbb{R}^n)$$

thus $[\Lambda : \Lambda'] = 1$, hence S_Λ generates X.

$\boxed{\xi}$

<u>Proof of the Compactness Criterion:</u>

(1) Assume $\overline{M}$ is compact.

Observe that $\mathcal{R}^{(n)} \longrightarrow \mathbb{N}$

$\Lambda \longmapsto \mathrm{Card}(\Lambda \cap \overline{B}(0,r)) \quad (\ast)$

is upper semicontinuous:

That is: If $\Lambda_n \longrightarrow \Lambda$ then

$$\limsup \mathrm{Card}(\Lambda_n \cap \bar{B}) \leq \mathrm{Card}(\Lambda \cap \bar{B})$$

This implies that (*) is bounded on $\bar{M}$ and hence there is $r' > 0$ with

$$\Lambda \cap B(0, r') = (0) \quad \forall \; x \in M.$$

(2) Let $M_r := \left\{ \Lambda \in \mathcal{R}^{(\ell)} : \Lambda \cap B(0,r) = (0) \right\}$

We indicate the main steps in the proof that M_r is rel. compact. Details are left to the reader:

Let Λ_n be a sequence in M_r.

Then $S_{\Lambda_n} := \Lambda_n \cap B(0, c(r))$ generates Λ_n

and

$$|S_{\Lambda_n}| \leq \frac{\mathrm{Vol}\left(B\left(0, c(r) + \frac{r}{2}\right)\right)}{\mathrm{Vol}\left(B\left(0, \frac{r}{2}\right)\right)}$$

since $\forall \; \lambda, \mu \in S_{\Lambda_n}$, if $\lambda \neq \mu$, $\|\lambda - \mu\| \geq \frac{r}{2}$.

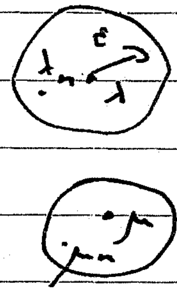
Λ_L

Thus there is a subsequence s.t. in the Hausdorff topology

$$S_{\lambda_n} \longrightarrow S$$

Where (1) Card S_n = Card S

(2) $\forall \varepsilon > 0$, $\forall \lambda \in S$ there is exactly one $\lambda_n \in S_{\lambda_n}$.



Then one shows easily that $\Lambda := \langle S \rangle$ is a lattice and $\Lambda_n \to \Lambda$. $\blacksquare$

3.2. Quaternion ~~Lattices~~ Algebras and Arithmetic lattices in $SL(2,\mathbb{R})$

Let K be a field and $a, b \in K^\times$.
On the 4-dimensional K-vector space

$$H_{a,b}(K) := \{ x_1 1 + x_2 \cdot i + x_3 j + x_4 k : x_i \in K \}$$

we put a multiplication defined on the basis $\{1, i, j, k\}$ by

$$ij = -ji = k, \quad i^2 = a, \quad j^2 = b,$$

with the requirment that K commutes with every basis element. Then:

Prop. 3.10 (1) This multiplication extends by bilinearity in a unique way to ~~an~~

an associative K-algebra structure on $H_{a,b}(K)$.

(2) The map $x \longrightarrow \bar{x}$,

$$\bar{x} := x_1 - x_2 i - x_3 j - x_4 k$$

is an involutory antiautomorphism of K.

(3) $N(x) := x\bar{x}$, gives a multiplicative map $N: H_{a,b}(K) \longrightarrow K$, that is $N(xy) = N(x)N(y)$

(4) x is invertible $\Longleftrightarrow N(x) \neq 0$ in which case $x^{-1} := \frac{\bar{x}}{N(x)}$.

Proof: (1) The requirment of associativity and that FC center, determines uniquely the product of basis elements: $ik = i(ij) = (ii)j = aj$

$ki = -jii = -aj$

$jk = -bi \qquad kj = ib$

etc....

The rest is a verification.

(2) Verify.

(3) $N(x) = (x_1 + x_2 i + x_3 j + x_4 k)(x_1 - x_2 i - x_3 j - x_4 k$

$= x_1^2 - x_2^2 a - x_3^2 b + x_4^2 ab$

$- x_1 (x_2 i + x_3 j + x_4 k)$

$+ x_1 (x_2 i + x_3 j + x_4 k)$

$+ x_2 x_3 (-ij - ji)$

$+ x_2 x_4 (-ik - ki) \quad + x_3 x_4 (-kj - jk)$

$= x_1^2 + x_2^2 a + x_3^2 b + x_4^2 ab.$

Thus $N(xy) = xy(\overline{xy})$

$= xy\bar{y}\bar{x} = N(y)N(x).$

(4) Say x is invertible: then $x x^{-1} = 1$

$\Rightarrow N(x) N(x^{-1}) = 1 \Rightarrow N(x) \neq 0.$

Conversely if $N(x) \neq 0$, then

$$\frac{\bar{x}}{N(x)} \in H_{a,b}(K) \quad \text{and.} \quad x\,\frac{\bar{x}}{N(x)} = \frac{N(x)}{N(x)}$$
$$= 1.$$

□

<u>Corollary 3.11</u>. $H_{a,b}(K)$ is a division algebra $\Longleftrightarrow N^{-1}(0) = 0$.

<u>Examples 3.12</u>.

(1) $K = \mathbb{R}$, $a = b = -1$, $\mathbb{H}_{-1,-1}(\mathbb{R})$ = Hamilton quaternions.

(2) $H_{1,b}(K) \simeq M_{2,2}(K)$ for any field K. Indeed:

$$\varphi(x) = \begin{pmatrix} x_1 + x_2 & (x_3 + x_4)b \\ x_3 - x_4 & x_1 - x_2 \end{pmatrix}$$

ives the desired isomorphism.

(3) Let $K = \mathbb{Q}$, b a prime and $a \in \mathbb{N}$ not a quadratic residue mod b.

Then $H_{a,b}(\mathbb{Q})$ is a division algebra.

Indeed assume N has a non-trivial zero. We may then assume $x_1, \dots, x_4 \in \mathbb{Z}$, $\gcd(x_1, \dots, x_4) = 1$ and

$$x_1^2 - a x_2^2 - b x_3^2 + ab x_4^2 = 0.$$

Reducing mod b we get

$$x_1^2 = a x_2^2 \quad \text{in } \mathbb{Z}/b\mathbb{Z}.$$

But since a is not a square mod b $\Longrightarrow$

$b \mid x_1$ and $b \mid x_2$, that is:

$x_1 = y_1 b$; $x_2 = y_2 b$ hence

$$y_1^2 b^2 - a b^2 y_2^2 - b x_3^2 + ab x_4^2 = 0$$

$$y_1^2 b - ab y_2^2 - x_3^2 + a x_4^2 = 0.$$

Thus $x_3^2 = a x_4^2$ hence as above $b \mid x_3$, $b \mid x_4$. Contradiction.

Now let $A \subset \mathbb{C}$ be any subring with 1, we set $H_{a,b}(A) := A1 + Ai + Aj + Ak$ an A-submodule of $H_{a,b}(\mathbb{C})$.

Observe that

$$H^1_{a,b}(A) := \{ x \in H_{a,b}(A) : N(x) = 1 \}$$

is a group.

<u>Example 3.13</u>: $H^1_{-1,-1}(\mathbb{R})$ is the 3-sphere $\simeq S^3$.

$$H^1_{-1,-1}(\mathbb{Z}) = \text{finite group}$$

with 8 elements

$$\{ \pm 1, \pm i, \pm j, \pm k \}.$$

Now we'll turn to the construction of arithmetic lattices. We assume for the remainder of this section that a, b are positive integers.

<u>Lemma 3.14</u> $a, b \in \mathbb{N}^*$. Then

$$h : H_{a,b}(\mathbb{R}) \longrightarrow M_{2,2}(\mathbb{R})$$

$$x \longmapsto \begin{pmatrix} x_1 + x_2\sqrt{a} & (x_3 + x_4\sqrt{a})b \\ x_3 - x_4\sqrt{a} & x_1 - x_2\sqrt{a} \end{pmatrix}$$

is an algebra isomorphism with

$$N(x) = \det h(x).$$

<u>Proof</u>: $1 \longmapsto Id$

$$i \longmapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}$$

$$j \longmapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & \sqrt{a}\, b \\ -\sqrt{a} & 0 \end{pmatrix}$$

In particular h induces Lie group isomorphisms:

$$h : H_{a,b}(\mathbb{R})^{\times} \longrightarrow GL(2,\mathbb{R})$$

$$h : H_{a,b}(\mathbb{R})^{1} \longrightarrow SL(2,\mathbb{R}).$$

Now $H_{a,b}(\mathbb{Z})^1 = \{ x \in x_1 + x_2 i + x_3 j + x_4 k$

$x_i \in \mathbb{Z},\ x_1^2 - a x_2^2 - b x_3^2 + ab x_4^2 = 1 \}$

being the intersection of $H^1_{a,b}(\mathbb{R})$ and $H_{ab}(\mathbb{Z})$ is a discrete subgroup of $H_{a,b}(\mathbb{R})^1$. Let $\Gamma_{a,b} := h(H_{a,b}(\mathbb{Z})^1) < SL(2,\mathbb{R})$

<u>Thm. 3.15</u> If $H_{a,b}(\mathbb{Q})$ is a division algebra, the homogeneous space

$$SL(2,\mathbb{R}) / \Gamma_{a,b}$$

is compact.

In particular $\Gamma_{a,b}$ is a lattice.

Need some preliminaries: every $x \in H_{a,b}(\mathbb{R})$ determines endomorphisms

$$L_x, R_x \in \mathrm{End}(H_{a,b}(\mathbb{R})) \text{ given}$$

resp. by left, right multiplication.

In the basis $\{1, i, j, k\}$ the matrix of L_x is

$$\begin{pmatrix} x_1 & x_2 b & x_3 b & -x_4 ab \\ x_2 & x_1 & x_4 b & -x_3 b \\ x_3 & -x_4 a & x_1 & x_2 a \\ x_4 & -x_3 & x_2 & x_1 \end{pmatrix}$$

and R_x:

$$\begin{pmatrix} x_1 & a x_2 & x_3 b & -ab x_4 \\ x_2 & x_1 & -x_4 b & x_3 b \\ x_3 & a x_4 & x_1 & -a x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}$$

And $\det L_x = \det R_x = N(x)^2$.

For $m \in \mathbb{Z}$, let

$$H_{a,b}(\mathbb{Z})^m = \{ x \in H_{a,b}(\mathbb{Z}) : N(x) = m \}.$$

Then the group $H_{a,b}(\mathbb{Z})^1$ acts on $H_{a,b}(\mathbb{Z})^m$ by left and right multiplication.

<u>Lemma 3.18</u> For $m \neq 0$ there are finitely many left resp. right $H_{a,b}(\mathbb{Z})^1$-orbits in $H_{a,b}(\mathbb{Z})^m$.

<u>Proof</u>: Since $H_{a,b}(\mathbb{Z})^m$ and $H_{a,b}(\mathbb{Z})^1$ are inv. under involution $x \mapsto \bar{x}$, it suffices the show the lemma for the right action.

For $x \in H_{a,b}(\mathbb{Z})^m$:

$$H_{a,b}(\mathbb{Z}) \cdot x = R_x(H_{a,b}(\mathbb{Z})) \subset H_{a,b}(\mathbb{Z})$$

are both lattices in $H_{a,b}(\mathbb{R})$; thus the index equals:

$$\left| H_{a,b}(\mathbb{Z}) \big/ H_{a,b}(\mathbb{Z})x \right| = |\det R_x| = m^2.$$

Now ~~let~~ there are finitely many subgroups of index m^2 in $H_{a,b}(\mathbb{Z}) \cong \mathbb{Z}^4$: indeed any such subgroup must contain $m^2\mathbb{Z}^4$ and $\left|\mathbb{Z}^4 \big/ m^2\mathbb{Z}^4\right| = m^8$.

Thus let

$$H_{a,b}(\mathbb{Z})x_1, \dots, H_{a,b}(\mathbb{Z})x_{r(m)}$$

be the set of distinct subgroups of $H_{a,b}(\mathbb{Z})$ so obtained. Then $\forall x \in H^m_{a,b}(\mathbb{Z})$ there is $1 \le i \le r(m)$ with

$$H_{a,b}(\mathbb{Z})x = H_{a,b}(\mathbb{Z})x_i$$

Setting $y = x_i x^{-1}$, we have that

$H_{a,b}(\mathbb{Z}) = H_{a,b}(\mathbb{Z})\cdot y$ hence $y \in H_{a,b}(\mathbb{Z})$

and $N(y)=1$, that is, $y \in H_{a,b}(\mathbb{Z})^1$.

Thus: $$H_{a,b}(\mathbb{Z})^m = \bigsqcup_{i=1}^{r(m)} H_{a,b}(\mathbb{Z})^1 x_i \qquad \blacksquare$$

Proof of Thm 3.5

We show that $H(\mathbb{R})^1 / H(\mathbb{Z})^1$ is compact.

Let $g \in H(\mathbb{R})^1$: then $\Lambda := L_g(H(\mathbb{Z}))$ is a lattice in $H(\mathbb{R})$ of covolume 1; hence the convex set

$$V = \{ y \in H(\mathbb{R}) : |y_i| \leq 1 \quad 1 \leq i \leq 4 \}$$

meets Λ in say gx, $x \in H(\mathbb{Z})$.

$\cancel{gx \in V \cap \Lambda}$

Since $H(\mathbb{Q})$ is a division algebra and $x \neq 0$, we have $N(x) =: m \in \mathbb{Z} \setminus \{0\}$.

Thus $m = N(x) = N(gx) \leq (a+1)(b+1)$

the last inequality following from

$gx \in V$.

Let now

$$V_m := \{ x \in H(\mathbb{R}) : |x_i| \leq 1 , N(x) = m \}.$$

Then V_m is a compact subset of $H(\mathbb{R})^*$

and we have $gx \in V_m$.

Applying lemma 3.14 to $x \in H(\mathbb{Z})^m$

there exists $x' \in H(\mathbb{Z})^1$ s.t. $x = x' x_i$

$(1 \leq i \leq r(m))$. Hence

$$g x' \in V_m x_i^{-1} \subset \bigcup_{\substack{1 \leq |m| \leq (a+1)(b+1) \\ 1 \leq i \leq r(m)}} V_m x_i^{-1}$$

Let $C := H(\mathbb{R})^1 \cap \left(\bigcup V_m x_i^{-1} \right)$

Then C is a compact subset of $H(\mathbb{R})^1$

and we have shown that $\forall g \in H(\mathbb{R})^1$

There is $x' \in H(\mathbb{Z})^1$ with $gx' \in G'$.

$\boxed{\xi}$