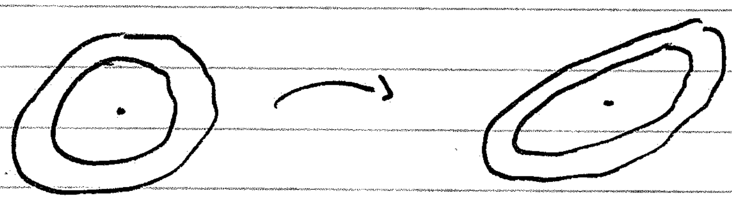
Chapter 4. Ergodic Actions: the Howe–Moore theorem.

Recall Mostow's rigidity theorem (see Introduction Thm 1.8) to the effect that if $M_1 = \Gamma_1 \backslash \mathbb{H}^n_{\mathbb{R}}$, $M_2 = \Gamma_2 \backslash \mathbb{H}^n_{\mathbb{R}}$ are compact quotients of real hyperbolic n-space and $f : M_1 \longrightarrow M_2$ is a homotopy equivalence then if $n \geq 3$, f is homotopic to an isometry. There are two key insights of Mostow in this proof. More precisely let $\theta : \Gamma_1 \longrightarrow \Gamma_2$ be the isomorphism induced by f on the level of fundamental groups and $\tilde{f} : \mathbb{H}^n_{\mathbb{R}} \longrightarrow \mathbb{H}^n_{\mathbb{R}}$ a lift of f.

The first insight is

(1) $\tilde{f}$ extends to a quasiconformal homeo $\Psi : \partial \mathbb{H}^n_{\mathbb{R}} \longrightarrow \partial \mathbb{H}^n_{\mathbb{R}}$ that is equivariant, namely $\Psi(\gamma \xi) = \theta(\gamma) \Psi(\xi)$ $\forall \xi \in \partial \mathbb{H}^n_{\mathbb{R}}$ and $\gamma \in \Gamma_1$.

Focus on $n = 3$: then $\partial \mathbb{H}^3_{\mathbb{R}}$ can be identified with the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ and under this identification $SO(3,1)^\circ$ acts like $PSL(2, \mathbb{C})$ namely by conformal maps. Intuitively, quasi-conformal means that Ψ maps circles at small scale to ellipses with uniformly bounded excentricity:



This idea of constructing boundary maps in a rigidity context can be considerably extended and this will be the subject of Chapter 5.
The second insight is:
(2) the action of Γ_1 on $\partial \mathbb{H}^n_{\mathbb{R}}$ is ergodic.

This is a form of transitivity in the measure theory context: it implies that if φ were not conformal, then there would be a Γ_1-invariant line field ~~of~~ on $\mathbb{P}^1(\mathbb{C})$ contradicting ergodicity. Thus φ is conformal and by a classical theorem of Liouville comes from an element of $PSL(2,\mathbb{C})$. This makes it also clear where the hypothesis $n \geq 3$ is used. Both properties (1) and (2) also hold for $n=2$. But the circle $S^1 = \partial \mathbb{H}^2_{\mathbb{R}}$ has just one line field and we gain no information in this case.

The object of this chapter is to ~~prove~~ establish the necessary tools on ergodic actions that are needed to establish the superrigidity theorem.

We place ourselves in the general context of a second countable locally compact group R acting continuously on a locally compact second countable topological space X, $R \times X \longrightarrow X$. Let μ be a positive R-quasiinvariant measure on X; recall that this means that

$$\mu(E) = 0 \iff \mu(gE) = 0 \quad \forall g \in R.$$

Def. 4.1. The R-action on (X, μ) is ergodic if for every R-invariant measurable subset $E \subset X$ we have either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

We start giving some examples, urging the reader to work out the details.

Example 4.2
Let G be a l.c. s.c. group, $H < G$ a closed subgroup and μ a quasiinvariant measure on G/H (see Chapt. 2.). Then the G-action on $(G/H, \mu)$ is ergodic.

Example 4.3.
Let $X = \mathbb{R}/\mathbb{Z}$ be the circle with its Haar measure μ and $\alpha \notin \mathbb{Q}/\mathbb{Z}$. Then the $\mathbb{Z}$ action on $\mathbb{R}/\mathbb{Z}$

$$\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$
$$(m, x) \longmapsto x + m\alpha$$

preserves μ and is ergodic.
When $\alpha \in \mathbb{Q}/\mathbb{Z}$ the action is not ergodic.
We shortly indicate the proof of ergodicity.
Recall that the Fourier transform of a function $f \in L^1(\mathbb{R}/\mathbb{Z}, \mu)$ is given by

$$\hat{f} : \mathbb{Z} \to \mathbb{C},$$
$$\hat{f}(n) = \int_{\mathbb{R}/\mathbb{Z}} f(x) e^{2\pi i n x} \, d\mu(x)$$

and a classical theorem says the $f \mapsto \hat{f}$ is injective. For $y \in \mathbb{R}/\mathbb{Z}$ set ${}_y f(x) = f(x+y)$ and let's compute

$$\widehat{{}_y f}(n) = \int_{\mathbb{R}/\mathbb{Z}} f(x+y) e^{2\pi i n x} \, d\mu(x)$$
$$= \int_{\mathbb{R}/\mathbb{Z}} f(x) e^{2\pi i n (x-y)} \, d\mu(x)$$

$$= e^{-2\pi i n y} \hat{f}(n).$$

If now $E \subset \mathbb{R}/\mathbb{Z}$ is invariant under $x \to x + \alpha$ then we get for $f = \chi_E$:

$$\widehat{f_{-\alpha}}(n) = \hat{f}(n) \quad \forall n \in \mathbb{Z}$$

$$= e^{2\pi i n \alpha} \hat{f}(n)$$

and since $\alpha \notin \mathbb{Q}/\mathbb{Z}$, $e^{2\pi i n \alpha} \neq 1 \;\; \forall n \in \mathbb{Z} \setminus \{0\}$

implying $\hat{f}(n) = 0 \quad \forall n \in \mathbb{Z} \setminus \{0\}$.

Thus $\hat{f} = \hat{f}(0) \cdot \delta_0$.

On the other hand the Fourier transform of $\mathbb{1}_{\mathbb{R}/\mathbb{Z}}$ is δ_0 and hence by injectivity of Fourier transform we get

$$\chi_E = f = \hat{f}(0)\, \mathbb{1}_{\mathbb{R}/\mathbb{Z}} = \mu(E)\, \mathbb{1}_{\mathbb{R}/\mathbb{Z}}$$

This equality is understood in $L^1(\mathbb{R}/\mathbb{Z}, \mu)$

which implies $\mu(E) = 0$ or $\mu(\mathbb{R}/\mathbb{Z} \setminus E) = 0$.

<u>Example 4.4.</u>

Let Γ be a countable group, $\{0,1\}$ with discrete topology and $X = \{0,1\}^\Gamma$ with product topology. Then Γ acts continuously on X by $(\gamma f)(\eta) = f(\gamma^{-1}\eta)$, $f \in \{0,1\}^\Gamma$.

Let μ be the product measure

$$\left(\tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_1\right)^\Gamma$$

Then μ is a Γ-invariant probability measure and the Γ-action is ergodic.

<u>Example 4.5</u>

Consider the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with its Haar measure μ. Then $SL(n,\mathbb{Z})$ acts on $\mathbb{T}^n$ by automorphisms leaving μ invariant and this action is ergodic.

Hint: proceed as in Example 4.3; use Fourier transform and the Riemann–Lebesgue theorem.

In rigidity theory one often encounters R-invariant measurable maps $f: X \to Y$ taking values in a space Y equipped with a σ-algebra $\mathcal{B}$ of subsets and, thinking about the analogy between transitive and ergodic actions, one would like to conclude that f is "essentially" constant, that is, there is $y \in Y$ such that

$$\mu(X \setminus f^{-1}(y)) = 0.$$

Let us henceforth call Borel space a set Y endowed with a σ-algebra $\mathcal{B}$ of subsets. For the above conclusion to hold

one clearly needs some hypothesis on $(Y, \mathcal{B})$: indeed take for instance Example 4.3, set $Y = \mathbb{Z}\backslash(\mathbb{R}/\mathbb{Z})$ with quotient topology and corresponding Borel structure. The canonical projection

$$p: \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{Z}\backslash(\mathbb{R}/\mathbb{Z})$$

is clearly $\mathbb{Z}$-invariant measurable but not essentially constant (Why?).

Before turning to this we have to get out of the way a bothersome issue namely the maps I alluded to are often constructed by using fixed point theorems for R-actions in function spaces that is classes of functions. We say that $f: X \to Y$ measurable is essentially invariant if for every $g \in R$ the maps $x \mapsto f(gx)$ and $x \mapsto f(x)$

coincide almost everywhere. In the following lemma we will use our hypotheses on (X, μ) and R to the effect that μ is σ-finite, the left Haar measure μ_R on R is σ-finite as well and Fubini's theorem applies to $(R \times X, \mu_R \times \mu)$.

<u>Lemma 4.6</u>. Let $(Y, \mathcal{B})$ be a Borel space and $f: X \to Y$ measurable essentially R-invariant. Then there exists

$$f_0 : X \to Y$$

measurable, R-invariant, coinciding with f almost everywhere.

<u>Proof</u>: For every $g \in R$, $f(g^{-1}x) = f(x)$ for a.e. $x \in X$. By Fubini this implies that for $\mu_R \times \mu$ a.e. $(g, x) \in R \times X$,

$$f(g^{-1}x) = f(x).$$

By Fubini again this implies that the set $X_1 = \{ x \in X :$ the map $\begin{array}{l} R \to Y \\ g \mapsto f(g^{-1}x) \end{array}$

coincides μ_R-a.e. with the constant map $g \mapsto f(x)$ $\}$ has complement of measure zero. Let us consider the potentially larger set

$$X_0 = \{ x \in X : R \to Y,\ g \mapsto f(g^{-1}x) \text{ is essentially constant} \}$$

Clearly $\mu(X \sim X_0) = 0$. Moreover since μ_R is left R-invariant, the set X_0 is R-invariant. Now define

$$f_0(x) = \begin{cases} \text{essential value of } g \mapsto f(g^{-1}x) & , x \in X_0 \\ y_0 & , x \notin X_0 \end{cases}$$

where $y_0 \in Y$ is an arbitrary but fixed point in Y. Then $f_0 : X \to Y$ is R-invariant

and coincides with f on X_1. $\blacksquare$

<u>Def. 4.7.</u> The Borel space $(Y, \mathcal{B})$ is countably separated if there is a countable family in $\mathcal{B}$ separating the points of Y.

<u>Example 4.8.</u> If Y is a second countable topological space then endowed with the σ-algebra of Borel sets, it is countably separated.

<u>Prop. 4.9</u> Assume $\mathcal{R} \times X \to X$ is ergodic and let $f: X \to Y$ be ~~a~~ measurable essentially $\mathcal{R}$-invariant measurable map with values in a countably separated Borel space $(Y, \mathcal{B})$. Then f is essentially constant.

Proof: Since X is a countable union of compact sets we may take a probability measure ν on X that is equivalent to μ. Also in view of lemma 4.6 we may assume that f is R-invariant. Let $\{A_i : i \in \mathbb{N}\} \subset \mathcal{B}$ be a countable family separating the points of Y.

Since f is invariant, ~~for~~ for every $p \in Y$ $f^{-1}(p)$ is R-invariant measurable hence $\nu(f^{-1}(p)) = 1$ or 0. Hence there is at most one point $p \in Y$ with $\nu(f^{-1}(p)) = 1$.

Let for every $p \in Y$ and $n \geq 1$,

$$X_p(n) = f^{-1}\left(\bigcap_{\substack{i=1 \\ p \in A_i}}^{n} A_i\right)$$

Since $\{A_i : i \in \mathbb{N}\}$ separates points we have $f^{-1}(p) = \bigcap_{n \geq 1} X_p(n)$.

We have $\nu(f^{-1}(p)) = \lim_{n \to \infty} \nu(X_p(n))$

and hence the sequence

$$\{\nu(X_p(n)) : n \geq 1\}$$

is stationnary. Thus $\forall p \in Y$ $\exists\, n(p) \geq 1$ with $\nu(f^{-1}(p)) = \nu(X_p(n_p))$.

Now $\{X_p(n_p) : p \in Y\}$ is a countable

family of measurable sets covering X, thus $\nu(X_p(n_p\delta)) > 0$ for some $p \in Y$ and we are alone. $\boxed{\Xi}$

Now we turn to the main theme of this chapter with the following question: let $\Gamma < G$ be a lattice in a l.c.s.c. group G and μ the G-invariant probability measure on $\Gamma \backslash G$. Then any closed subgroup $R < G$ acts on $\Gamma \backslash G$ preserving μ. The question is, when is this action ergodic?

<u>Exercise 4.10</u> Show that if R is compact, then R acts ergodically on $\Gamma \backslash G$ iff R acts transitively.

The main result of this chapter concerning the above question is:

Theorem 4.11. Let G be a connected simple Lie group with finite center and $\Gamma < G$ a lattice. Then any closed non-compact subgroup $H \leq G$ acts ergodically on $\Gamma \backslash G$.

This result will be a consequence of a fundamental theorem of Howe and Moore concerning continuous unitary representations of G.

We now introduce the relevant objects in their natural degree of generality.

Let G be locally compact, say s.c.,

Def. 4.12. A unitary representation of G into a Hilbert space $\mathcal{H}$ is a group homomorphism $\pi: G \longrightarrow U(\mathcal{H})$ into

the group of unitary operators $U(\mathcal{H})$ of $\mathcal{H}$. It is called continuous if the action map

$$G \times \mathcal{H} \longrightarrow \mathcal{H}$$
$$(g, v) \longmapsto \pi(g)v$$

is continuous.

The fundamental example of such an object is

Example 4.13

Let

$$G \times X \to X$$
$$(g, x) \longmapsto x.g$$

be a continuous right action of G on a l.c. s.c. space X and assume μ is a G-invariant positive Radon measure. Define for $g \in G$ and $f \in L^2(X, \mu)$

$$(\lambda(g) f)(x) = f(xg).$$

Using the G-invariance of μ it is readily seen that $\|\lambda(g)f\|_2 = \|f\|_2 \ \forall g \in G$ $\forall f \in L^2(X,\mu)$. For the continuity one establishes first that for every $f \in C_{00}(X)$

$$\begin{aligned} G &\longmapsto C_{00}(X) \\ g &\longmapsto \lambda(g)f \end{aligned}$$

is continuous for $\| \ \|_2$ on $C_{00}(X)$ which then by density of $C_{00}(X)$ extends to $L^2(X,\mu)$.
Then one makes the observation that a unitary representation is continuous iff $\forall v \in \mathcal{H}$,

$$\begin{aligned} G &\longrightarrow \mathcal{H} \\ g &\longmapsto \pi(g)v \end{aligned}$$

is continuous.

<u>Def. 4.17</u>: every pair of vectors $u, v \in \mathcal{H}$ give rise to a function $G \longrightarrow \mathbb{C}$, $g \mapsto \langle \pi(g)u, v \rangle$ called matrix coefficient of π. It is a continuous

if $\tilde{\pi}$ is.

Here is the central theorem on unitary representations:

<u>Thm 4.15</u> (Howe–Moore). Let G be connected simple with finite center. Let $(\pi, \mathcal{H})$ be a continuous unitary representation into a separable Hilbert space $\mathcal{H}$ such that the subspace of G-fixed vectors

$$\mathcal{H}^G := \{ v \in \mathcal{H} : \pi(g)v = v \ \forall g \in G \}$$

is reduced to (0). Then all matrix coefficients of π are functions on G that vanish at infinity.

<u>Exercise 4.16</u> Show that this thm fails for $G = \mathbb{R}$ and more generally for G

connected solvable. Show that it also fails for $G = \widetilde{SL(2,\mathbb{R})}$; the latter is a simple Lie group but it has infinite center.

<u>Proof that Thm 4.15 $\Rightarrow$ Thm 4.11.</u>

Let μ be the G-invariant probability measure on $\Gamma\backslash G$ and consider

$$\lambda : G \to U\,L^2(\Gamma\backslash G, \mu)$$

the continuous unitary representation of G in $L^2(\Gamma\backslash G)$ defined by $\lambda(g)f(x) = f(xg)$. Which is a special case of Example 4.13.

We are going to compute $L^2(\Gamma\backslash G)^R$ the space of R-invariant vectors. To this end observe that $\mathbb{1}_{\Gamma\backslash G} \in L^2(\Gamma\backslash G)$ and since G acts transitively on $\Gamma\backslash G$

$$L^2(\Gamma\backslash G)^G = \mathbb{C}\cdot\mathbb{1}_{\Gamma\backslash G}.$$

Let $\mathcal{H}_0 := \{ f \in L^2(\Gamma\backslash G) : \langle f, \mathbb{1}_{\Gamma\backslash G} \rangle = 0 \}$.
Then $\mathcal{H}$ is a G-invariant closed subspace of $L^2(\Gamma\backslash G)$ and we have the orthogonal decomposition: $L^2(\Gamma\backslash G) = \mathbb{C}\,\mathbb{1}_{\Gamma\backslash G} \oplus \mathcal{H}_0$

and hence
$$L^2(\Gamma\backslash G)^R = \mathbb{C}\,\mathbb{1}_{\Gamma\backslash G} \oplus \mathcal{H}_0^R.$$

Now if $\lambda_0(g) := \lambda(g)\big|_{\mathcal{H}_0}$, $g \in G$

we obtain a continuous unitary representation of G into $\mathcal{H}_0$ for which evidently
$$\mathcal{H}_0^G = (0).$$
Thus if $f \in \mathcal{H}_0^R$ then we have on one hand $\langle \lambda_0(g) f, f \rangle = \langle f, f \rangle \; \forall g \in R$ and on the other hand since R is not compact we can find a sequence $(g_n)_{n \geq 1}$ in R leaving every compact subset of G which implies by Thm 4.15

$$\lim_{n\to\infty} \langle \lambda_0(g_n) f, f \rangle = 0$$

and hence $\langle f, f \rangle = 0$, that is $f = 0$.

Thus we have shown that

$$L^2(\Gamma\backslash G)^R = \mathbb{C}\,\mathbb{1}_{\Gamma\backslash G}.$$

If now $E \subset \Gamma\backslash G$ is a measurable R-inv. subset then $\chi_E \in L^2(\Gamma\backslash G)^R = \mathbb{C}\,\mathbb{1}_{\Gamma\backslash G}$

which implies $\mu(E) = 0$ or $\mu(\Gamma\backslash G \setminus E) = 0$.

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