We will present the proof of the Howe-Moore theorem for $G = SL(n,\mathbb{R})$; the case of general simple Lie groups does not require any new ideas.

Before explaining the strategy we will make a first reduction of the problem.
This ~~uses~~ the Cartan decomposition

$$G = K A^+ K$$

where $K = SO(n)$ and

$$A^+ = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \begin{array}{l} \lambda_1 \geq \lambda_2 \cdots \geq \lambda_n > 0 \\ \prod_{i=1}^{n} \lambda_i = 1 \end{array} \right\}$$

<u>Lemma 4.17</u> Let $\pi : G \to U(\mathcal{H})$ be a continuous unitary representation. The following are equivalent:

(1) all coefficients of π vanish at infinity.

(2) all coefficients of π vanish at infinity along A^+.

<u>Proof:</u> (1) $\Rightarrow$ (2) clear.

(2) $\Longrightarrow$ (1) :

Assume there are $u, v \in \mathcal{H}$ and a sequence $(g_n)_{n\geq 1}$ in G with $\lim_{n\to\infty} g_n = +\infty$ and

$$\limsup_{n\to\infty} |\langle \pi(g_n)u, v\rangle| > 0.$$

Let $g_n = k_n a_n k_n'$ with k_n, k_n' in $SO(n)$ and $a_n \in A^+$. Passing to a subsequence we may assume

$$\begin{cases} \lim_{n\to\infty} |\langle \pi(g_n)u, v\rangle| > 0 \\ \lim k_n = k \\ \lim k_n' = k' \end{cases}$$

Then $\langle \pi(g_n)u, v\rangle = \langle \pi(a_n)\pi(k_n')u, \pi(k_n)^{-1}v\rangle$

Thus:

$$|\langle \pi(g_n)u, v\rangle - \langle \pi(a_n)\pi(k')u, \pi(k)^{-1}v\rangle|$$

$$= |\langle \pi(a_n)\pi(k'_n)u, \pi(k_n)^{-1}v\rangle - \langle \pi(a_n)\pi(k')u, \pi(k)^{-1}v\rangle|$$

$$= |\langle \pi(a_n)(\pi(k'_n)u - \pi(k')u), \pi(k_n)^{-1}v\rangle$$

$$+ \langle \pi(a_n)\pi(k')u, \pi(k_n)^{-1}v - \pi(k)^{-1}v\rangle|$$

$$\leq \|\pi(k'_n)u - \pi(k')u\| + \|\pi(k_n)^{-1}v - \pi(k)^{-1}v\|$$

and hence with $u' = \pi(k')u$, $v' = \pi(k)^{-1}v$:

$$\lim_{n\to\infty} |\langle \pi(g_n)u, v\rangle - \langle \pi(a_n)u', v'\rangle| = 0$$

which implies $\lim_{n\to\infty} |\langle \pi(a_n)u', v'\rangle| > 0$

Since $g_n \to \infty$ in G and K is compact this implies $\lim a_n = +\infty$ in A^+ which proves (2) $\Rightarrow$ (1) and hence the lemma. $\boxed{\xi}$

Thus we want to show that under the hypothesis of the Howe-Moore theorem for every $u, v \in \mathcal{H}$ and $(a_n)_{n \geq 1}$ in A^+ with $\lim a_n = \infty$,

$$\lim_{n \to \infty} \langle \pi(a_n) u, v \rangle = 0.$$

In this context it will be useful to introduce the concept of ultraweak convergence on the space $\mathcal{L}(\mathcal{H})$ of bounded operators in $\mathcal{H}$. We say that a sequence $(T_n)_{n \geq 1}$ in $\mathcal{L}(\mathcal{H})$ convergences ultraweakly to $T \in \mathcal{L}(\mathcal{H})$ if $\forall u, v \in \mathcal{H}$

$$\lim_{n \to \infty} \langle T_n u, v \rangle = \langle T u, v \rangle.$$

The next lemma shows that for $(\pi(a_n))_{n \geq 1}$ as above, there is always a subsequence converging ultraweakly; the proof of Howe-

Moore consists then in studying such ultraweak limits and eventually show that they must be zero.

<u>Lemma 4.18</u> If $(T_n)_{n\geq 1}$ is a sequence in $\mathcal{L}(\mathcal{H})$ with $\|T_n\| \leq 1$ $\forall n \geq 1$ and $\mathcal{H}$ is separable, there is a subsequence converging ultraweakly.

<u>Proof</u>:

Let $e_1, e_2, \dots, e_k, \dots$ an orthonormal basis of $\mathcal{H}$ so in particular $\bigoplus_{k\geq 1} \mathbb{C} e_k$ is dense in $\mathcal{H}$.

For every $k, \ell \geq 1$ the sequence $(\langle T_n e_k, e_\ell \rangle)_{n\geq 1}$ is contained in $\{ z \in \mathbb{C} : |z| \leq 1 \}$.

By Kantor diagonal argument there exists a subsequence $(n_j)_{j\geq 1}$ such that

for every $k, l \geq 1$, $\left(\langle T_{n_j} e_k, e_l\rangle\right)_{j\geq 1}$
converges. This implies that
$$\left(\langle T_{n_j} u, v\rangle\right)_{j\geq 1}$$
converges for every $u, v \in \bigoplus_{k\geq 1} \mathbb{K} e_k$;
from this and the density of $\bigoplus_{k\geq 1} \mathbb{K} e_k$
one deduces easily that
$$\left(\langle T_{n_j} u, v\rangle\right)_{j\geq 1}$$
is a Cauchy sequence $\forall u, v \in \mathcal{H}$.
Define then $c(u, v) = \lim_{j\to\infty} \langle T_{n_j} u, v\rangle$

Then $c : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a sesquilinear
form with $|c(u,v)| \leq \|u\|\|v\|$ $\forall u, v \in \mathcal{H}$
and hence there is $T \in \mathcal{L}(\mathcal{H})$, with
$$c(u,v) = \langle Tu, v\rangle \quad \forall u, v \in \mathcal{H}$$
which shows the lemma. $\square$

If $S \subset G$ is a subset, let

$$\mathcal{H}^S = \{ v \in \mathcal{H} : \pi(g) v = v \; : \forall g \in S \}$$

which is a closed subspace; if $\langle S \rangle < G$ is the subgroup generated by S then

$$\mathcal{H}^S = \mathcal{H}^{\langle S \rangle}.$$

We will also need to take the orthogonal of a subset $\mathcal{L} \subset \mathcal{H}$, namely

$$\mathcal{L}^\perp = \{ v \in \mathcal{H} : \langle v, u \rangle = 0 \; \forall u \in \mathcal{L} \}$$

which is a closed vector subspace. In the sequel we will need the following elementary properties

(4.19) If $\mathcal{L} \subset \mathcal{H}$ is a closed vector subspace then $\mathcal{L}^{\perp\perp} = \mathcal{L}$

(4.20) If A, B are vector subspaces of $\mathcal{H}$

(4.20 bis) If A, B are closed vector subspaces then

$$\overline{A^\perp + B^\perp} = (A \cap B)^\perp$$

Indeed from (4.20) we get by replacing $A^\perp$ and $B^\perp$ by A, B:

$A \cap B = (A^\perp + B^\perp)^\perp$. Now since

$(A^\perp + B^\perp)^\perp = \overline{A^\perp + B^\perp}^{\perp}$ we get by using 4.19:

$$(A \cap B)^\perp = \overline{A^\perp + B^\perp}.$$

$$(A+B)^{\perp} = (A^{\perp} \cap B^{\perp}).$$

<u>Lemma 4.21.</u> Let $S \subset G$ be a subset. Then

$$\overline{\operatorname{Lin}\{\pi(s)w - w\}}_{\substack{w \in \mathcal{H} \\ s \in S}} = (\mathcal{H}^S)^{\perp}.$$

<u>Proof</u>: Clearly, for $w \in \mathcal{H}$ and $v \in \mathcal{H}^S$:

$\langle \pi(s)w - w, v \rangle = 0$ which shows the inclusion $\subset$. Applying the $\perp$ to this inclusion and using (4.19) we get:

$$\mathcal{H}^S \subset \overline{\operatorname{Lin}\{\pi(s)w - w : \substack{w \in \mathcal{H} \\ s \in S}\}}^{\perp}.$$

Let's prove the opposite inclusion: if $\xi \in \overline{\operatorname{Lin}\{\pi(s)w - w : \substack{w \in \mathcal{H} \\ s \in S}\}}^{\perp}$ then

$$\langle \xi, \pi(s)w - w \rangle = 0 \quad \forall w \in \mathcal{H}, \forall s \in S$$

hence $\langle \pi(s)^{-1}\xi, w \rangle = \langle \xi, w \rangle \quad \forall w \in \mathcal{H}$

and thus $\xi = \pi(s)\xi \quad \forall s \in S$.

This shows the equality

$$\mathcal{H}^S = \overline{\operatorname{Lin}\left\{\pi(s)w - w : \begin{array}{l} w \in \mathcal{H} \\ s \in S \end{array}\right\}}^{\perp}$$

and hence the lemma by (4.19). $\boxed{\xi}$

The next lemma reveals a crucial idea in the proof of Howe-Moore.

<u>Lemma 4.22</u> Let $(a_j)_{j\geq 1}$ be a sequence in A^+ such that $(\pi(a_j))_{j\geq 1}$ converges ultra weakly with limit E. Let

$$\mathcal{U}^+ := \left\{ h \in G : \lim_{j\to\infty} a_j^{-1} h a_j = e \right\}$$

$$\mathcal{U}^- := \left\{ h \in G : \lim_{j\to\infty} a_j h a_j^{-1} = e \right\}$$

Then

$$\left(\mathcal{H}^{\langle \mathcal{U}^+, \mathcal{U}^- \rangle}\right)^{\perp} \subset \operatorname{Ker} E.$$

<u>Remark 4.22 bis</u>: It is instructive to see how this already implies Howe-Moore for $SL(2,\mathbb{R})$. Indeed: if $(a_j)_{j\geq 1}$ is any sequence in A^+ with $\lim_{j\to\infty} a_j = +\infty$, then $a_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j^{-1} \end{pmatrix}$

$\lambda_j > 1$ and $\lim_{j\to\infty} \lambda_j = +\infty$. But then

$$\mathcal{U}^+ = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$\mathcal{U}^- = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

and $\langle \mathcal{U}^+, \mathcal{U}^- \rangle = SL(2,\mathbb{R})$. Thus if E is any ultraweak limit of any subsequence of $(\pi(a_j))_{j\geq 1}$ we have

$$\left(\mathcal{H}^{SL(2,\mathbb{R})}\right)^\perp \subset \operatorname{Ker} E \quad \text{and thus}$$

if $\mathcal{H}^{SL(2,\mathbb{R})} = (0)$, $E = 0$.

Proof:

Let $h \in \mathcal{U}^-$ and $u, v \in \mathcal{H}$:

$$\langle E\pi(h)u, v\rangle = \lim_{j\to\infty} \langle \pi(a_j)\pi(h)u, v\rangle$$

$$= \lim_{j\to\infty} \langle \pi(a_j h a_j^{-1})\pi(a_j)u, v\rangle.$$

But now: $\quad = \lim_{j\to\infty} \langle \pi(a_j)u, \pi(a_j h a_j^{-1})^{-1} v\rangle$

$$\left| \langle \pi(a_j)u, \pi(a_j h a_j^{-1})^{-1} v\rangle - \langle \pi(a_j)u, v\rangle \right|$$

$$\leq \| \pi(a_j h a_j^{-1})^{-1} v - v \| \longrightarrow 0 \text{ for } j \to \infty.$$

Thus

$$\langle E\pi(h)u, v\rangle = \langle Eu, v\rangle$$

that is $\quad \mathrm{Lin}\{\pi(h)u - u : u \in \mathcal{H}, h \in \mathcal{U}^-\}$

$$\subset \operatorname{Ker} E$$

and hence by lemma 4.21:

$$\left(\mathcal{H}^{\mathcal{U}^-}\right)^{\perp} \subset \operatorname{Ker} E.$$

Likewise for E^* the adjoint of $\mathbb{E}$ we have

$$\langle E^* u, v \rangle = \langle u, E v \rangle = \lim_j \langle u, \pi(a_j) v \rangle$$

$$= \lim_j \langle \pi(a_j^{-1}) u, v \rangle$$

and by the same argument:

$$(\mathcal{H}^{u^+})^\perp \subset \operatorname{Ker} E^*.$$

Observe now that since A is abelian:

$$\langle \pi(a_j) u, \pi(a_k) v \rangle = \langle \pi(a_k^{-1}) u, \pi(a_j^{-1}) v \rangle$$

which implies by letting $j \to +\infty$ and then $k \to \infty$:

$$\langle E u, E v \rangle = \langle E^* u, E^* v \rangle$$

in particular $\|E u\|^2 = \|E^* u\|^2$,

hence $\operatorname{Ker} E = \operatorname{Ker} E^*$.

Thus $(\mathcal{H}^{u^-})^\perp + (\mathcal{H}^{u^+})^\perp \subset \operatorname{Ker} \mathbb{E}$

and thus by using 4.20 bis:

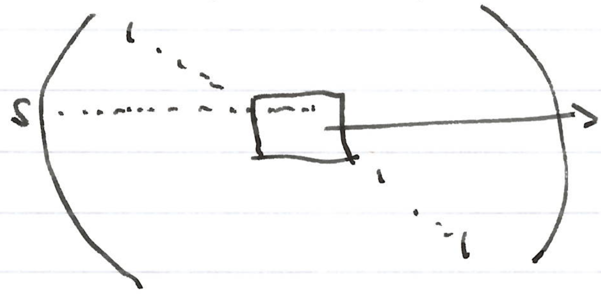
$$\left(\mathcal{H}^{\langle u^-, u^+\rangle}\right)^{\perp} = \overline{\left(\mathcal{H}^{u^-} \cap \mathcal{H}^{u^+}\right)^{\perp}}$$

$$= \left(\mathcal{H}^{u^-}\right)^{\perp} + \left(\mathcal{H}^{u^+}\right)^{\perp} \subset \operatorname{Ker} E. \quad \boxtimes$$

For $G = SL(n, \mathbb{R})$,
given remark 4.22 (bis) we need to gain some information on U^+, U^- for sequences $(a_\ell)_{\ell \geq 1}$ in A^+ with $\lim_{l \to \infty} a_\ell = +\infty$.

<u>Lemma 4.23.</u> Let $G = SL(n, \mathbb{R})$ and $(a_\ell)_{\ell \geq 1}$ a sequence in A^+ with $\lim_{l \to \infty} a_\ell = +\infty$. Then there is a subsequence $(a_{n_\ell})_{\ell \geq 1}$ and an index $1 \leq s \leq n-1$ such that setting $a'_\ell := a_{n_\ell}$, the corresponding subgroups

u^+, u^- have the property that

$\langle u^+, u^- \rangle \supset$  bloc $SL(2,\mathbb{R})$

<u>Proof</u>: The proof is an elaboration on 4.22 (bis).

Let $a_j = \mathrm{diag}(\lambda_1^{(j)}, \dots, \lambda_n^{(j)})$

with $\lambda_1^{(j)} \geq \dots \geq \lambda_n^{(j)} > 0$, $\prod_{k=1}^{n} \lambda_k^{(j)} = 1$.

Since $a_j \to +\infty$ this implies $\lim_{j \to \infty} \lambda_1^{(j)} = +\infty$.

Now write

$$\lambda_1^{(j)} = \lambda_1^{(j)} / \lambda_2^{(j)} \cdots\cdots \lambda_{n-1}^{(j)} / \lambda_n^{(j)} \cdot \lambda_n^{(j)}$$

$$< \prod_{k=1}^{n-1} \lambda_k^{(j)} / \lambda_{k+1}^{(j)}$$

Then there exists $1 \leq s \leq n-1$

and a subsequence $(n_\ell)_{\ell \geq 1}$

with $\displaystyle\lim_{\ell\to\infty} \lambda_s^{(n_\ell)} \Big/ \lambda_{s+1}^{(n_\ell)} = +\infty .$

Now it is a verification that

U^+ contains
$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & t & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

and U^- contains
$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & 0 & & \\ & & x & 1 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix}, \quad x \in \mathbb{R} .$$

$\blacksquare$

We present the next lemma in the context of simple Lie groups because the proof in the general case is just much more transparent.

<u>Lemma 4.24.</u> Let $\pi : G \to U(\mathcal{H})$ be a continuous unitary representation of G connected simple Lie group with finite center and $t \in G$ such that $\mathrm{Ad}(t) \in GL(\mathfrak{g})$ is diagonalisable with positive real eigenvalues not all $= 1$. Then

$$\mathcal{H}^{\langle t \rangle} = \mathcal{H}^{G}.$$

<u>Remark 4.25</u> The relevant example for $G = SL(n, \mathbb{R})$ is

$$t = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \boxed{\begin{matrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{matrix}} & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \qquad \lambda > 1.$$

<u>Proof</u>: Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$

the direct sum decomposition of $\mathfrak{g}$ where

$\mathfrak{g}_-$ = sum of the $Ad(t)$ - eigenspaces with eigenvalue < 1

$\mathfrak{g}_0$ = subspace of $Ad(t)$-fixed vectors

$\mathfrak{g}_+$ = sum of the $Ad(t)$ - eigenspaces with eigenvalue > 1.

Claim: $\mathcal{H}^{\langle t \rangle} \subset \mathcal{H}^{\langle \exp \mathfrak{g}_- \rangle}$.

Let $u \in \mathcal{H}^{\langle t \rangle}$ and $X \in \mathfrak{g}_-$:

$$\langle \pi(\exp X) u, u \rangle = \langle \pi(\exp X)\pi(t^{-\ell})u, \pi(t^{-\ell})u \rangle$$

$$= \langle \pi(t^{\ell}(\exp X)t^{-\ell})u, u \rangle$$

$$= \langle \pi(\exp \mathrm{Ad}(t)^{\ell} X)u, u \rangle$$

Since $\lim\limits_{\ell \to \infty} \mathrm{Ad}(t)^{\ell} X = 0$ we get

$$\langle \pi(\exp X)u, u \rangle = \langle u, u \rangle$$

and hence $\pi(\exp X)u = u$ which proves the claim.

Replacing $\mathfrak{g}_-$ by $\mathfrak{g}_+$ and taking the above limit for $\ell \to -\infty$ gives

$$\mathcal{H}^{\langle t \rangle} \subset \mathcal{H}^{\langle \exp \mathfrak{g}_+ \rangle}.$$

Now observe that every element of $\exp \mathfrak{g}_0$ commutes with t and hence

$\mathcal{H}^{\langle t \rangle}$ is invariant under $\langle \exp \mathfrak{g}_0 \rangle$.

Now by Lie theory we know that the

image of $\quad \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ \longrightarrow G$

$$(X, Y, Z) \longmapsto \exp X \exp Y \exp Z$$

contains a neighborhood $\mathcal{U}$ of $e \in G$.
Thus $\mathcal{H}^{\langle t \rangle}$ is $\mathcal{U}$-invariant and hence
$G = \langle \mathcal{U} \rangle$-invariant. Consider then

$$\pi_0 : \quad G \longrightarrow U(\mathcal{H}^{\langle t \rangle})$$
$$g \longmapsto \pi(g)\Big|_{\mathcal{H}^{\langle t \rangle}}.$$

Obviously $\operatorname{Ker} \pi_0 \supset \langle t \rangle$, but
t cannot be in the center of G and
since $\mathfrak{g}$ is simple this implies $\operatorname{Ker} \pi_0 = G$
hence $\mathcal{H}^{\langle t \rangle} = \mathcal{H}^G$. $\boxed{\;}$

<u>Remark 4.26.</u> In the case $G = SL(n, \mathbb{R})$ it is well known that any $N \triangleleft G$ either coincides with G or is contained in $Z(G)$ which is either $\{+Id\}$ if n is odd or $\{\pm Id\}$ if n is even. Thus $t \notin Z(G)$, and any normal subgroup containing t is equal to G.

<u>Proof of Howe-Moore (Thm 4.15)</u>

Let $\pi : G \longrightarrow U(\mathcal{H})$ be a continuous unitary representation in a separable Hilbert space $\mathcal{H}$ with $\mathcal{H}^G = 0$.

~~Assume some coefficient of π does not vanish at ∞; then some coefficient does not vanish at ∞ along A^+ (Lemma 4.17). Say there is $(a_\ell)_{\ell \geq 1}$ in A^+~~

Let $(a_\ell)_{\ell \geq 1}$ be a sequence in A^+ such that $\lim_{\ell \to \infty} a_\ell = +\infty$ and $(\pi(a_\ell))_{\ell \geq 1}$ converges ultraweakly to $E \in \mathcal{L}(\mathcal{H})$.

We are going to show that $E = 0$.

Passing to a subsequence we may assume that (lemma 4.23) that

$$\langle u^+, u^- \rangle \supset \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \boxed{SL(2,\mathbb{R})} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

and hence

$$\langle u^+, u^- \rangle \ni t = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad \lambda > 1.$$

From this we deduce

$$\mathcal{H}^{\langle u^+, u^- \rangle} \subset \mathcal{H}^{\langle t \rangle}$$

and hence: $\left(\mathcal{H}^{\langle t\rangle}\right)^{\perp} \subset \left(\mathcal{H}^{\langle u^+, u^-\rangle}\right)^{\perp}$

By lemma (4.22) and lemma 4.24 this implies

$$\mathcal{H} = (0)^{\perp} = \left(\mathcal{H}^{G}\right)^{\perp} = \left(\mathcal{H}^{\langle t\rangle}\right)^{\perp} \subset \left(\mathcal{H}^{\langle u^+, u^-\rangle}\right)^{\perp} \subset \operatorname{Ker} E$$

and hence $E = 0$.

If now $a \to \langle \pi(a)u, v\rangle$ does not vanish at ∞ along A^+ then there is $(a_\ell)_{\ell \geq 1}$ with $\lim a_\ell = +\infty$ and

$|\langle \pi(a_\ell)u, v\rangle| \geq \varepsilon > 0 \quad \forall \ell \geq 1.$

By lemma 4.18 we may assume (passing to a subsequence) that $(\pi(a_\ell))_{\ell \geq 1}$ converges ultra weakly to some $E \in \mathcal{L}(\mathcal{H})$ which by the above implies $E = 0$, a contradiction since $|\langle \pi(a_\ell)u, v\rangle| \geq \varepsilon > 0$ $\forall \ell \geq 1$. Hence every coefficient of π vanishes at ∞ along A^+ and hence by lemma 4.17 along G. $\boxed{}$

We close this chapter by mentionning a consequence of Thm 4.11 often used in rigidity arguments, namely

<u>Corollary 4.27</u> Let G be simple connected with finite center, $\Gamma < G$ a lattice and $R < G$ closed non-compact. Then the Γ-action on G/R is ergodic.

This follows easily from the following, which we leave as an exercise; it is a straightforward consequence of Thm 2.20.

Lemma 4.28 Let G be l.c. second countable and H_1, H_2 closed subgroups of G. Then H_1 acts ergodically on G/H_2 iff H_2 acts ergodically on $H_1\backslash G$.

Interlude: Amenable Groups.

In this short interlude we introduce the concept of amenability and develop the tools necessary for the applications in Chapter 5 to the existence of boundary maps.

Recall that a Freshet space is a topological Vector space E that is Hausdorff and whose topology is defined by a family of semi-norms $\| \ \|_\alpha$, $\alpha \in I$.

Definition I-1: A topological group L is amenable if for every continuous linear action $L \times E \to E$ on a Freshet space and any convex, non-empty L-invariant compact subset $G \subset E$ there is a L-fixed point. We say that the group L is amenable if L endowed with discrete topology is amenable.

Prop. I - 2 :

(1) Any abelian group is amenable.

(2) Any extension of topological groups

$$\{e\} \to A \to R \to B \to \{e\}$$

where A and B are amenable, is amenable.

(3) Any solvable group is amenable.

(4) Any compact topological group is amenable.

Proof:

(1) Let $A \times E \to E$ be an ~~affine~~ action of an abelian group by continuous linear maps of E and $G \subset E$ A-invariant,

convex, compact, non-empty. For $g \in A$ let $E^g = \{ v \in E : g(v) = v \}$ and $C^g = C \cap E^g$.

For $y \in C$ and $n \in \mathbb{N}$, the convex combination $C_n(y) := \frac{1}{n+1} [y + gy + \cdots + g^n y]$

is in C. Moreover for any semi-norm $\| \ \|_\alpha$ on E we have:

$$\| g C_n(y) - C_n(y) \|_\alpha \leq \frac{2}{n+1} \max_{x \in C} \| x \|_\alpha .$$

Therefore any cluster point of the sequence $(C_n(y))_{n \geq 1}$ is g-fixed and hence $C^g \neq \phi$. Since A is abelian it leaves E^g invariant and leaves $C^g \subset E^g$ in it invariant as well. Thus $\forall g, h \in A : (C^g)^h = C^g \cap C^h \neq \phi$; by recurrence $\bigcap_{g \in F} C^g \neq \phi \quad \forall F \subset A$ finite and by compactness, $\bigcap_{\mathfrak{J} \in A} C^{\mathfrak{J}} \neq \phi$. $\natural$

(2) Assume $R \times E \to E$ and $C \subset E$ are as in the definition of amenability. Since A is amenable we deduce

$$\phi \neq C^A \subset E^A.$$

Since $A \triangleleft R$, R acts continuously on E^A, and this action factors via $R/A \simeq B$. Since B is amenable we deduce $(C^A)^B \neq \phi$ and hence $C^R \neq \phi$. $\natural$

(3) Clear from (1) and (2).

(4) Let $K \times E \to E$ and C as in the definition of amenability. Let μ_K be the probability Haar measure on K.

Pick $v \in G'$ and let $\nu \in M^1(G')$ be the direct image of μ_K under

$$\begin{aligned} K &\longrightarrow G' \\ k &\longmapsto k_* v . \end{aligned}$$

Then the center of mass $c(\nu) \in G'$ is a K-fixed point. $\boxed{}$

<u>Exercise I-3.</u> Assume L is an amenable topological group. Then for any continuous action $L \times M \to M$ on a compact Hausdorff topological space, there exists an L-invariant probability measure on M.

Here is the fact we were after:

<u>Corollary I-4.</u> Let G be simple, connected, finite center and $P < G$ minimal parabolic. Then P is an amenable topological group.

<u>Example I–5</u> : $G = SL(n, \mathbb{R})$

$$\underline{P} = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \right\}.$$

In this case $\underline{P}$ is solvable hence amenable.

<u>Proof</u>: In general $\underline{P} = M \cdot R$ where $R \lhd \underline{P}$ is connected solvable and M is compact. The assertion follows then from (2), (3), (4) of Prop. I – 2. $\boxed{\xi}$