Chapter 5. Lattices and boundary maps.

This is the central chapter of this course. In it we will establish the existence and uniqueness of boundary maps under very general conditions.

Before turning to the description of the results (section 5.2) and their proofs in subsequent sections, I begin in Section 5.1 by giving a short overview of Furstenberg's boundary theory. While these results are not needed in the formal development of the chapter, they underline where the concepts and ideas of proofs are coming from and thus are essential for the conceptual understanding of this chapter.

5. 1. Boundary Theory.

We begin with a very classical theorem on harmonic functions. Let

$$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$$

be the unit disc in $\mathbb{C}$. Recall that $f : \mathbb{D} \longrightarrow \mathbb{C}$ is harmonic if it is C^2 and $\Delta f \equiv 0$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator.

For $\zeta \in \partial\mathbb{D} := S^1$ and $z \in \mathbb{D}$,

$$P(z, \zeta) := \frac{1 - |z|^2}{|z - \zeta|^2}$$

is the Poisson kernel, and for every $\zeta \in S^1$, $z \longrightarrow P(z, \zeta)$ is harmonic.

Let $\mathcal{L}$ be the usual Lebesgue probability measure on S^1; for $f \in L^1(S^1, \mathcal{L})$

its Poisson transform $Pf : \mathbb{D} \to \mathbb{C}$ is defined by

$$Pf(z) = \int_{S^1} f(\zeta)\, P(z,\zeta)\, d\alpha(\zeta).$$

The question of representability of harmonic functions on $\mathbb{D}$ by Poisson integrals has a particularly simple answer in the case of bounded harmonic functions. Namely, let $\mathcal{H}^\infty(\mathbb{D})$ be the space of bounded harmonic functions on $\mathbb{D}$ endowed with the sup-norm:

<u>Theorem 5.1.</u> The Poisson transform induces an isometric isomorphism

$$P : L^\infty(S^1) \to \mathcal{H}^\infty(\mathbb{D})$$

of Banach spaces.

Now we turn to Furstenberg's interpretation of this result in terms of analysis on the group $G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1, \ a, b \in \mathbb{C} \right\}$

It is well known that $SU(1,1)$ acts by Möbius transformations on $\mathbb{D}$:

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} z = \frac{az + b}{\bar{b}z + \bar{a}}$$

and in this way the quotient $PSU(1,1)$ by the center of $SU(1,1)$ acts effectively and coincides with $Is(\mathbb{D})^0$ the connected component of the group of isometries of $\mathbb{D}$ endowed with its Poincaré metric:

$$ds^2 = \frac{|dz|^2}{(1 - |z|^2)^2}.$$

Given $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1,1)$

let's compute $\int_{S^1} f(g\zeta)\, d\mathcal{L}(\zeta)$ by

change of variable $\eta = g\zeta$ we get

$$d\mathcal{L}(\zeta) = |(g^{-1})'(\eta)|\, d\mathcal{L}(\eta)$$

$$= \frac{1}{|-\bar{b}\eta + a|^2}$$

On the other hand:

$$P(g(0), \eta) = \frac{1 - \left|\frac{b}{\bar{a}}\right|^2}{\left|\frac{b}{\bar{a}} - \eta\right|^2} = \frac{1}{|b - \bar{a}\eta|^2}$$

$$= \frac{1}{|-\bar{b}\eta + a|^2}$$

and thus

$$\int_{S^1} f(g\zeta)\, d\mathcal{L}(\zeta) = \int_{S^1} f(\eta)\, P(g(0), \eta)\, d\mathcal{L}(\eta)$$

$$= (Pf)(g(0)).$$

Thus we can interpret the Poisson transform as a map

$$\mathcal{P} : L^\infty(S^1) \longrightarrow C_h(G)$$

$$\mathcal{P}f(g) = \int_{S^1} f(gy)\, d\mathcal{L}(y).$$

The next observation is that all functions in the image of $\mathcal{P}$ satisfy mean value properties and that this is solely connected to the properties of the measure $\mathcal{L}$ wrt the G-action on S^1.

For $\mu \in M^1(G)$ probability measure and $F : G \to \mathbb{C}$ a Borel function define

$$(F * \mu)(g) = \int_G F(gh)\, d\mu(h).$$

Let $K = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \bar{\xi} \end{pmatrix} : |\xi|^2 = 1 \right\} < G$ and μ any left K-invariant probability measure on G.

We compute

$$Pf * \mu(g) = \int_G \int_{S^1} f(gh\eta)\, d\alpha(\eta)\, d\mu(h)$$

This can be written in the following way: let $G \times S^1 \to S^1$, $(g,\eta) \mapsto g\eta$ be the action map; let $\mu * \alpha$ denote the direct image of $\mu \times \alpha$ via this action map: this is a probability measure on S^1 and

$$(Pf * \mu)(g) = \int_{S^1} f(g\xi)\, d(\mu * \alpha)(\xi).$$

Thus let's compute $\mu * \alpha$: for $\varphi \in C(S^1)$,

$$(\mu * \alpha)(\varphi) = \int_G \int_{S^1} \varphi(h\eta)\, d\alpha(\eta)\, d\mu(h).$$

We have

$$= \int_G \int_{S^1} \varphi(\eta) P(h(o), \eta)\, d\alpha(\eta)\, d\mu(h)$$

$$= \int_{S^1} \varphi(\eta) \int_G P(h(o), \eta)\, d\mu(h).$$

Now: μ is left K-invariant hence

$$\int_G P(h(0),\eta)\,d\mu(h) = \int_K \int_G P(kh(0),\eta)\,d\mu(h)\,d\lambda_K(k)$$

where λ_K is the probability Haar measure on K.

But: $$P(kh(0),\eta) = \frac{1-|h(0)|^2}{|kh(0)-\eta|^2}$$

$$= \frac{1-|h(0)|^2}{|h(0)-k^{-1}\eta|^2}$$

$$= P(h(0),k^{-1}\eta).$$

Thus: $$\int_G P(h(0),\eta)\,d\mu(h) = \int_G d\mu(h) \int_K P(h(0),k^{-1}\eta)\,d\lambda_K(k)$$

But now $$\int_K P(h(0),k^{-1}\eta)\,d\lambda_K(k)$$

$$= \int_{S^1} P(h(0),\xi)\,d\alpha(\xi)$$

$$= (\mathcal{P}\mathbb{1}_{S^1})(h) \equiv 1.$$

This shows that $\mu * \mathcal{L} = \mathcal{L}$, which is a property of $\mathcal{L}$ refered to as μ-stationnary. And this leads to

$$\mathcal{P}f * \mu = \mathcal{P}f$$

which we refer to as μ-harmonic.

Let then $\mathcal{H}^\infty_\mu(G)$ be the space of bounded μ-harmonic functions on G.

Furstenbergs generalization of the Poisson representation takes the following form

<u>Thm 5.2.</u> Let G be connected, semisimple, finite center $K < G$ maximal compact $P < G$ minimal parabolic and let ν_K be the K-invariant probability measure on G/P and $\mu \in M'(G)$ any left K-inv. probability measure that is absolutely

continuous wrt the Haar measure on G.

Then the Poisson transform

$$\mathcal{P} : L^\infty(G/_P, \nu_K) \longrightarrow \mathcal{H}^\infty_\mu(G)$$

is a Banach space isomorphism.

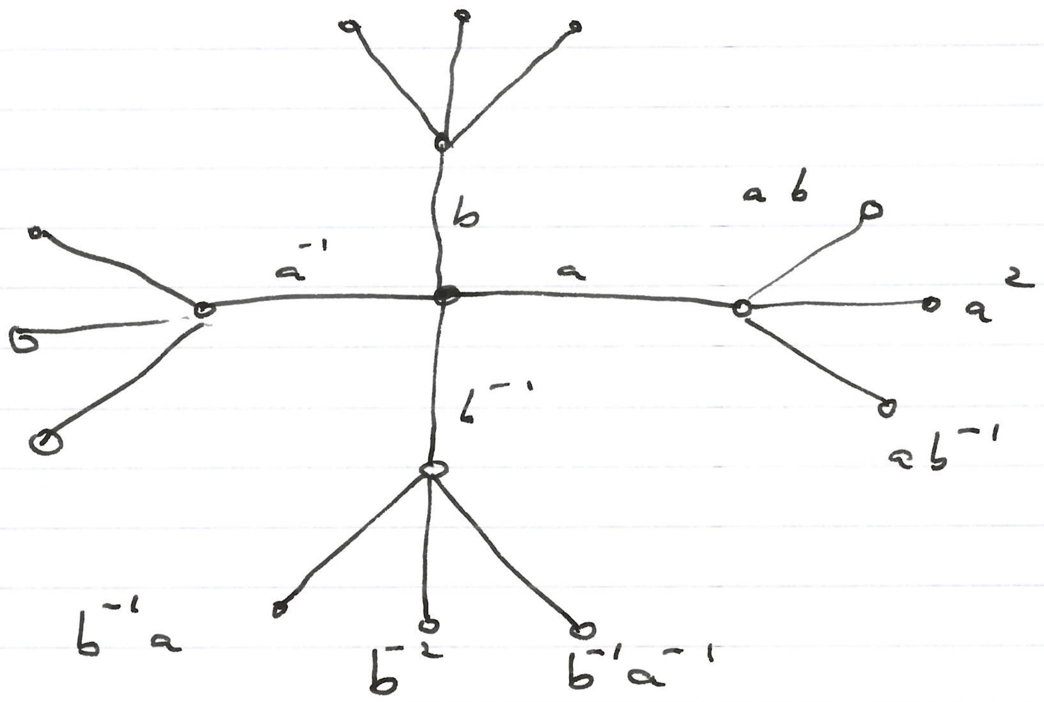
Incidentally this also says that in our case $G = SU(1,1)$ and for such measures μ, $\mathcal{H}^\infty_\mu(G)$ coincides with the space $\mathcal{H}^\infty(\mathbb{D})$ of harmonic bounded functions on $\mathbb{D}$; to make the connection observe that since μ is left K-invariant, any μ-harmonic function is right K-invariant, and $G/_K = \mathbb{D}$.

Thm 5.2 can be found in H. Furstenberg, "A Poisson formula for semi-simple Lie groups" Annals of Math. 1963, Vol. 77, No. 2.

There it is also shown that in the context of Thm 5.2, if $X = G/K$ is the associated symmetric space, $\mathcal{H}_\mu^\infty(G)$ coincides with the space of bounded harmonic functions on X, where now harmonic refers to the Laplacian associated to the G-invariant riemannian metric on X.

Now Furstenberg's point of view allows to consider μ-harmonic functions on any locally compact group G, where μ is a probability measure on G.
For example let Γ be a finitely generated group with finite generating set $S = S^{-1}$. Consider $\mathrm{Cay}(\Gamma, S)$ the Cayley graph of Γ wrt S: its set of vertices is Γ and $\{\eta, \gamma\} \subset \Gamma$ are

adjacent, $\gamma \sim \eta$, if $\eta = \gamma s$ for some $S \ni s$. For instance if $\Gamma = \mathbb{F}(a,b)$ is the free group on two generators then with $S = \{a, b, a^{-1}, b^{-1}\}$, $\mathrm{Cay}(\Gamma, S)$ looks like:



Then a function $f : \Gamma \longrightarrow \mathbb{C}$ is μ-harmonic for $\mu = \frac{1}{|S|} \sum_{s \in S} \delta_s$

$\Longleftrightarrow$ $f : \Gamma \longrightarrow \mathbb{C}$, $\Gamma = \mathrm{Vertices}(\mathrm{Cay}(\Gamma, S))$

satisfies the mean value property:

$$f(\gamma) = \frac{1}{|S|} \sum_{\eta \sim \gamma} f(\eta).$$

In this degree of generality the question is whether μ-harmonic functions on a l.c. group G can be represented as Poisson transforms in some way to be made precise. This is the case and was shown by Furstenberg in "Random walks and discrete subgroups of Lie groups" in "Advances in Probability and related topics" Vol 1. 1973.

<u>Thm 5.3</u> Let G be l.c.s.c., and $\mu \in M^1(G)$ absolutely continuous wrt the left Haar measure and such that supp μ generates G. Then there is a standard Borel space B with a Borel action $G \times B \to B$ and a μ-stationary probability measure ν on B such that

$$\mathcal{P} : L^\infty(B,\nu) \longrightarrow \mathcal{H}^\infty_\mu(G)$$

$$\mathcal{P}f(g) = \int_B f(g\xi)\,d\nu(\xi)$$

is an isometric isomorphism.

The condition that μ is absolutely continuous ensures that μ-harmonic functions are automatically continuous, while the condition that supp μ generates G ensures that the measure ν is quasiinvariant. Such a space (B, ν) is called a Poisson boundary of (G, μ).
Since Furstenberg there has developped a big activity to identify Poisson boundaries for large classes of, mostly discrete, groups. Surveys can be found in:

M. Babillot: "An introduction to Poisson boundaries of Lie Groups"
in "Probability measures on groups: recent directions and trends"

Tata Institute Fund. Res. Mumbai 2006

A. Erschler: "Poisson-Furstenberg boundaries, large-scale Geometry and growth of groups"
Proc. of ICM, Vol. II,
pp 681 – 704, 2010.

At this point one can ask the question, what does this have to do with rigidity questions? This connection is established by the following result of Furstenberg which we state ~~~~ in the special case of semisimple groups. We place ourselves in the context of Thm V.2 which says that $(G/P, \nu_K)$ is the Poisson boundary of (G, μ).

<u>Theorem 5.4.</u> (Furstenberg in "Random walks at...")
Let $\Gamma < G$ be a lattice. Then there is a probability measure π on Γ (with $\operatorname{supp} \pi = \Gamma$) such that the associated Poisson boundary is $(G/P, \nu_K)$.

This is a truly astonishing theorem; it says that bounded harmonic functions on the symmetric space $X = G/K$ can be characterized by a mean value property of their restriction to Γ. Its application to rigidity is that it says that for a suitable probability measure on Γ we can recover $(G/P, \nu_K)$ as a Γ-space. Now it turns out that this in itself did not lead to a proof of Mostow rigidity for instance, but it has been recognized later that the underlying techniques could be used to establish the first step in the proof of Margulis superrigidity, and this first step is the topic of this chapter.

5.2. Description of the results.

This is the central chapter of this course. In it we will establish the existence and uniqueness of boundary maps under very general conditions.

In order to motivate the main theorem we begin by describing an existence theorem for boundary maps due to Furstenberg.

Let G be l.c. second countable (l.c.s.c.) $\Gamma < G$ a discrete subgroup ~~and $P < G$ a closed amenable subgroup~~ Assume $\Gamma \times X \to X$ is a continuous action on a compact metrizable space. Then Γ acts by isometries on $C(X)$ the Banach space of continuous functions with sup norm;

Hence it acts isometrically on its dual $C(X)^*$ which we identify with the space $M(X)$ of bounded measures endowed with the total variation norm. The group Γ acts also continuously on $C(X)^* \simeq M(X)$ endowed with the weak $*$-topology and preserves the compact convex subset $M^1(X)$ of probability measures. We will always consider $M^1(X)$ with its weak-$*$-topology for which it is metrisable since $C(X)$ is separable.

<u>Thm 5.5</u> Assume $P < G$ is a closed amenable subgroup. Then there exists

$$\varphi : G/P \to M^1(X)$$

Γ-equivariant measurable.

Thus in our situation, if now

$$\rho : \Gamma \longrightarrow GL(V)$$

is a representation where V is a f.d. vector space over a local field k, then Γ acts continuously on the compact metrisable space $\mathbb{P}(V)$ and thus there exists

$$\varphi : G/\underline{P} \longrightarrow M^1(\mathbb{P}V)$$

Γ-equivariant measurable.

The main theorem is then

Thm 5.6 Assume $\Gamma < G$ is a lattice in a l.c. compactly generated group G, $\underline{P} < G$ a closed amenable subgroup with $G/\underline{P}$ compact.

Let $\rho : \Gamma \to GL(V)$ be strongly irreducible, proximal. Then any Γ-equivariant measurable map $\varphi : G/\underline{P} \longrightarrow M^1(\mathbb{P}V)$

takes values in the subset of Dirac measures. In particular there is a unique Γ-equiv. measurable map $G/P \to P(V)$.

The proof relies on two different non-trivial ingredients. Both require the central concept of stationary measure.

Let quite generally L be a l.c.s.c group acting continuously

$$L \times M \to M$$

on a l-c. ~~metriz~~ metrisable space M and let $\mu \in M^1(L)$ be a probability measure. Given $\nu \in M^1(M)$ we define $\mu * \nu \in M^1(M)$ as the direct image of $\mu \times \nu$ under the product map $L \times M \to M$. In formulas:

$$(\mu * \nu)(f) = \int_L d\mu(l) \int_M d\nu(m) f(lm)$$

$f \in C(M)$.

Def. 5.7. ν is μ-stationnary if $\mu * \nu = \nu$.

While in general there is no L-invariant probability measure on M, stationnary measures exist under very general hypothesis.

Prop. 5.8 Let $L \times M \to M$ be a continuous action where L is l.c.s.c. and M is compact metrisable. Then for every $\mu \in M^1(L)$ there are μ-stationnary measures on M.

Proof: For every $n \geq 0$ define the continuous linear map $A_n(\mu) : M(M) \circlearrowleft$ by

$$A_n(\mu)\nu = \frac{1}{n+1} \sum_{j=0}^{n} \mu^{*j} * \nu$$

where $\mu^{*j} := \underbrace{\mu * \cdots * \mu}_{j \text{ times}}$.

and let S be the semigroup generated by $\{A_n(\mu) : n \geq 0\}$. Then $\forall\, T \in S$ we have $T(M'(M)) \subset M'(M)$ and $T(M'(M))$ is compact. We claim that

$$\bigcap_{T \in S} T(M'(M)) \neq \phi.$$

For this it is sufficient to show that for every finite collection $T_1, \ldots, T_k$:

$$\bigcap_{i=1}^{k} T_i(M'(M)) \neq \phi .$$

Let $T = T_1 \cdots T_k$: since all the T_i's commute we have

$$\phi \neq T(M'(M)) \subset T_i(M'(M)), \quad 1 \leq i \leq k$$

hence the claim. Let now

$$\nu_0 \in \bigcap_{T \in S} T(M'(M)).$$

Pick $n \geq 0$ and $\nu_n \in M^1(M)$ with

$$\nu_0 = A_n(\mu)\nu_n .$$

Then $\mu * \nu_0 - \nu_0 = \dfrac{\mu^{*n+1} * \nu_n - \nu_n}{n+1}$

and hence $\forall f \in C(M)$:

$$|(\mu * \nu_0 - \nu_0)(f)| \leq \frac{2\|f\|_\infty}{n+1}$$

which shows $\mu * \nu_0 = \nu_0$. $\boxed{\S}$

Of course nothing prevents one of taking

$$\mu = \delta_e$$

in which case not much information is gained. The really interesting measures for us are the admissible ones.

__Def. 5.~~8~~9.__ Let Γ be discrete. A probability measure $\mu \in M^1(\Gamma)$ is admissible if supp μ generates Γ as semigroup, that is

$$\bigcup_{k \geq 1} (\operatorname{supp} \mu)^k = \Gamma .$$

<u>Lemma 5.10.</u> Let $\Gamma \times M \to M$ act continuously on a l.c. metrizable space M, let $\mu \in M^1(\Gamma)$ be admissible and $\nu \in M^1(M)$ μ-stationary.

Then (1) $\operatorname{supp} \nu$ is Γ-inv.

(2) the measure class of ν is Γ-invariant.

<u>Proof</u>: From $\mu^{*k} * \nu = \nu$ we deduce

$\forall f \in C(M)$:

$$\sum_{\gamma} \mu^{*k}(\gamma) \int_M d\nu(m) f(\gamma m) = \int_M d\nu(m) f(m)$$

Hence if $\mu^{*k}(\eta) > 0$ and $f \geq 0$:

$$\mu^{*k}(\eta) \int_M d\nu(m) f(\eta m) \leq \int_M d\nu(m) f(m) \leq$$

$$\frac{1}{\mu^{*k}(\gamma)} \int_M d\nu(m) f(\gamma m).$$

But this inequality implies that $E \subset M$ is of ν-measure zero $\Leftrightarrow$ ~~...~~ $\gamma^{-1}(E)$ is of ν-measure zero

which shows (2), since $\operatorname{supp} \mu^{*k} = (\operatorname{supp} \mu)^k$.

(1) uses the same inequality and is left as an exercise. $\boxed{\text{W}}$

Theorem 5.8 will be a consequence of two major theorems involving stationary measures.

<u>Theorem 5.11.</u> Let Γ be a discrete group acting continuously on a locally compact metrisable space B, $\mu \in M^1(\Gamma)$ an admissible prob. measure and $\nu \in M^1(B)$

μ-stationary.

Let $\rho : \Gamma \to GL(V)$ be strongly irreducible acting proximaly on $\mathbb{P}(V)$. Then any Γ-equivariant ν-measurable map

$$\varphi : B \longrightarrow M^1(\mathbb{P}V)$$

takes values in the subset of Dirac measures.

The way one wishes to apply Thm 5.11 in order to prove Thm 5.6 is by picking $\mu \in M^1(\Gamma)$ admissible and setting $B = G/P$; since G/P is compact we have from Prop. 5.8 that there exists $\nu \in M^1(G/P)$, which is μ-stationary. However we only know that the measure class of ν is Γ-invariant but we would like ν to be in the G-invariant measure class. We are going to take advantage of the fact that in Thm 5.~~?~~11. we have the freedom of choice for $\mu \in M^1(\Gamma)$, as long as it is admissible.

The idea goes as follows.

Let μ_G be a left Haar measure on G and let $\psi \in C_{00}(G)$ satisfy the following conditions

$$(C) \quad \psi \geq 0, \quad \mu_0 := \psi \cdot \mu_G \in M^1(G)$$

and

$$\{ g \in G : \psi(g) > 0 \text{ and } \psi(g^{-1}) > 0 \}$$

generates G.

Observe that in the context of Thm 5.6 such a ψ exists since G is compactly generated. We have the following lemma whose proof is elementary and deffered for later:

<u>Lemma 5.12.</u> Assume G/P compact. Then there exists $\psi_0 \in C_{00}(G)$ satisfying condition (C) and such that whenever $\nu \in M^1(G/P)$ is μ_0-stationnary, ν is in the G-invariant measure class of G/P.

The second major theorem is

<u>Thm 5.13.</u> Let G be l.c.s.c. compactly generated, $P < G$ closed with G/P compact and $\mu_0 = \psi \cdot \mu_G$ with $\psi \in C_{00}(G)$ satisfying (C). Let $\nu_0 \in M^1(G/P)$ with $\mu_0 * \nu_0 = \nu_0$. Then there is $\mu \in M^1(\Gamma)$ with $\operatorname{supp}\mu = \Gamma$ such that $\mu * \nu_0 = \nu_0$.

Now we can show the mechanism that leads to a proof of Thm 5.6:

<u>Proof of Thm 5.6</u>:

Let $\mu_0 = \psi_0 \cdot m_G$ satisfying lemma 5.12. Since G/P is compact there exists $\nu_0 \in M^1(G/P)$, $\mu_0 * \nu_0 = \nu_0$; by lemma 5.8, ν_0 is in the G-invariant measure class of G/P. By Thm 5.13 there is $\mu \in M^1(\Gamma)$, $\operatorname{supp}\mu = \Gamma$ with $\mu * \nu_0 = \nu_0$. In particular μ is admissible. Now apply Thm 5.11 to $(B, \nu_0) = (G/P, \nu_0)$.

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