Percolation Theory - Exercise Sheet 1

Exercise 1.1. $^{(\star)}$

(a) Prove that $\{A \longleftrightarrow B\}$ is measurable for $A, B \subseteq \mathbb{Z}^d$.

(b) Prove that

$$\begin{aligned} F : \{0,1\}^E &\longrightarrow \mathbb{N} \cup \{+\infty\} \\ \omega &\longmapsto |C_x(\omega)| \end{aligned}$$

is measurable.

Exercise 1.2. $^{(\star)}$ An *infinite open path* from 0 (in ω) is a sequence $(\gamma_i)_{i\in\mathbb{N}}$ of distinct vertices such that $\gamma_0 = 0$ and for all $i \geq 1$, $\gamma_{i-1} \sim \gamma_i$ and $\omega(\gamma_{i-1}\gamma_i) = 1$. Prove that the event

$$A = \big\{\exists \text{ an infinite open path from } 0\big\}$$

is measurable.

Exercise 1.3. $^{(\star)}$

(a) Derive explicit expressions for $\mathrm{P}_p[|C_0| = 0]$, $\mathrm{P}_p[|C_0| \geq 1]$, and $\mathrm{P}_p[|C_0| = 1]$. Are these probabilities monotone in $p \in [0, 1]$?

(b) Calculate $\mathrm{P}_p[|C_0| \geq 1 | |C_x| = 0]$ for any $x \in \mathbb{Z}^d$.

Exercise 1.4. On $\mathbb{Z}^2$, consider the event

$$C_{2n,n} = \big\{\exists \text{ an open path from left to right in } [0, 2n] \times [0, n]\big\},$$

where we write $[0, 2n] \times [0, n]$ for the subgraph with vertex set $[0, 2n] \times [0, n] \cap \mathbb{Z}^2$ and edge set $\big([0, 2n] \times [0, n] \cap E\big) \setminus \big([0, 2n] \times \{n\} \cap E\big)$. For $q_n = 1 - \mathrm{P}_p[C_{2n,n}]$, prove that one of the following holds:

(i) $\exists c > 0$ such that $\forall n, q_n \geq c$,

(ii) $\exists c > 0$ such that $\forall n, q_n \leq e^{-cn}$.

Based on this result, prove that $p_c < 1$.