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HS2020

Percolation Theory - Exercise Sheet 2

Exercise 2.1.$^{(\star)}$ Assume that the event A is measurable with respect to $(\omega_j)_{j\neq i}$. Prove that

$$\mathrm{P}_{p_1,\ldots,p_k}[A] = \mathrm{P}_{p_1,\ldots,p_{i-1},0,p_{i+1},\ldots,p_k}[A],$$

which particularly implies that $\mathrm{P}_{p_1,\ldots,p_k}[A]$ does not depend on p_i. Here, we define $\mathrm{P}_{p_1,\ldots,p_k}[\{\omega\}] := \prod_{j=1}^k {p_j}^{\omega_j}(1-p_j)^{1-\omega_j}$ as in the proof of Russo's formula.

Exercise 2.2. Let $x, y \in \mathbb{Z}^d$ such that $x \neq y$. The goal of this exercise is to prove that $f(p) := \mathrm{P}_p[x \longleftrightarrow y]$ is *strictly* increasing in p.

(a) Use the monotone coupling to prove this.

(b) Use Russo's formula to prove this.

Exercise 2.3.$^{(\star)}$ Consider percolation on $\mathbb{Z}^d$.

(a) Let A, B be two decreasing events. Prove that

$$\mathrm{P}_p[A \cap B] \geq \mathrm{P}_p[A] \cdot \mathrm{P}_p[B].$$

(b) Let A be an increasing event, B a decreasing event. Prove that

$$\mathrm{P}_p[A \cap B] \leq \mathrm{P}_p[A] \cdot \mathrm{P}_p[B].$$

(c) [*Square Root Trick*] Let $k \geq 2$, $\epsilon > 0$. Let $A_1, \ldots, A_k$ be k increasing events. Assume that

$$\mathrm{P}_p\Big[\bigcup_{1\leq i\leq k} A_i\Big] \geq 1-\epsilon.$$

Prove that

$$\max_{1\leq i\leq k} \mathrm{P}_p[A_i] \geq 1-\epsilon^{\frac{1}{k}}.$$

Exercise 2.4. Consider percolation on $\mathbb{Z}^2$, and assume that

$$\mathrm{P}_p\big[\exists \text{ an open path in } \Lambda_n \text{ from left to right}\big] \xrightarrow{n\to\infty} 1.$$

(a) Show that there exist two sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ with $\frac{b_n-a_n}{n} \xrightarrow{n\to\infty} 0$ such that

$$\mathrm{P}_p\big[\exists \text{ an open path in } \Lambda_n \text{ from left to } \{n\}\times\{a_n,\ldots,b_n\}\big] \xrightarrow{n\to\infty} 1.$$

(b) Based on part (a), show that there exists a sequence $(c_n)_{n\geq 1}$ with $\frac{c_n}{n} \xrightarrow{n\to\infty} 0$ such that

$$\mathrm{P}_p\big[\exists \text{ an open path in } \Lambda_n \text{ from left to } \{n\}\times\{0,\ldots,c_n\}\big] \xrightarrow{n\to\infty} 1.$$