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Percolation Theory - Exercise Sheet 3

Exercise 3.1.$^{(\star)}$ Let $G = (V, E)$ be a graph. Consider two edges $e, f \in E$, and prove that

$$\mathrm{P}_p\left[\{e \text{ open}\} \circ \{f \text{ open}\}\right] \leq \mathrm{P}_p[e \text{ open}] \cdot \mathrm{P}_p[f \text{ open}] \leq \mathrm{P}_p\left[\{e \text{ open}\} \cap \{f \text{ open}\}\right].$$

Remark: Prove it directly, i.e. do not apply BK-Reimer inequality or FKG inequality.

Exercise 3.2.$^{(\star)}$ Let $G = (V, E)$ be a finite graph, and let $A \subseteq \{0,1\}^E$ be an increasing event, $B \subseteq \{0,1\}^E$ a decreasing event.

(a) Let $\omega \in A$. Prove that there exists a witness I for A in ω such that for all $e \in I$, $\omega(e) = 1$.

(b) Prove that $A \circ B = A \cap B$. Deduce that $\mathrm{P}_p[A \circ B] \leq \mathrm{P}_p[A] \cdot \mathrm{P}_p[B]$ in this case.

Exercise 3.3.$^{(\star)}$ Use the BK-Reimer inequality to prove that that for $x, y, z \in \mathbb{Z}^d$,

$$\mathrm{P}_p[\{x \longleftrightarrow y\} \circ \{y \longleftrightarrow z\}] \leq \mathrm{P}_p[\{x \longleftrightarrow y\}] \cdot \mathrm{P}_p[\{y \longleftrightarrow z\}].$$

Warning: The event $\{x \longleftrightarrow y\}$ depends on infinitely many edges, so one cannot directly apply the BK-Reimer inequality.

Exercise 3.4. Define the random variable

$$X_n = \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{1}_{x \longleftrightarrow \infty}.$$

Prove that

$$\lim_{n \to \infty} X_n = \theta(p). \qquad \text{(in probability)}$$

Hint: Look at the expectation and the variance of X_n.