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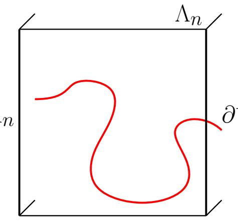
Percolation Theory - Exercise Sheet 4

Exercise 4.1.$^{(\star)}$ Let $p < p_c$. Define

$$\partial^-\Lambda_n = \{x \in \Lambda_n : x_1 = -n\},\ \partial^+\Lambda_n = \{x \in \Lambda_n : x_1 = n\},$$

which are two opposite sides of the boundary of $\Lambda_n = \{-n, \dots, n\}^d$.
Prove that

$$\lim_{n\to\infty} \mathrm{P}_p\Big[\,\partial^-\Lambda_n$$


$$\partial^+\Lambda_n\,\Big] = 0,$$

where the drawing represents an open path in Λ_n from $\partial^-\Lambda_n$ to $\partial^+\Lambda_n$.

Exercise 4.2.$^{(\star)}$

(a) Let $p < p_c$. Prove that $\mathbb{E}_p[|C_0|] < \infty$.

(b) Prove that $\mathbb{E}_{p_c}[|C_0|] = +\infty$.

(c) Show that $p_c(d) \geq \frac{1}{2d}$.

Hint for (b) and (c): First argue that $\forall S \subset \mathbb{Z}^d$ finite with $0 \in S$, $\phi_{p_c}(S) \geq 1$.

Exercise 4.3. [Fekete's lemma]
Let $(u_n)_{n\geq 1}$ be a sequence of numbers in $[-\infty, \infty)$ satisfying

$$u_{m+n} \leq u_m + u_n \qquad \text{(subadditivity)}$$

for all $m, n \geq 1$. Prove that the limit of $\left(\frac{u_n}{n}\right)$ exists in $[-\infty, \infty)$ and that

$$\lim_{n\to\infty} \frac{u_n}{n} = \inf_{n\geq 1} \frac{u_n}{n}.$$

Exercise 4.4. [Percolation with long-range interactions]
Let $G = (\mathbb{Z}^d, E)$ be the (complete) graph with vertex set $\mathbb{Z}^d$ and edge set

$$E := \left\{\{x, y\} : x, y \in \mathbb{Z}^d\right\},$$

and let $(J_{x,y})_{x,y\in\mathbb{Z}^d}$ be a family of non-negative, translation-invariant numbers, i.e. $J_{x,y} \geq 0$ and $J_{x,y} = J_{x+z,y+z}$ for all $x, y, z \in \mathbb{Z}^d$. We consider the bond percolation measure $\mathrm{P}_\beta, \beta \geq 0$, that is defined as the product measure on $\{0, 1\}^E$ (equipped with the product-σ-algebra) such that

$$\mathrm{P}_\beta[\{x, y\} \text{ is open}] = 1 - e^{-\beta J_{x,y}}, \ \mathrm{P}_\beta[\{x, y\} \text{ is closed}] = e^{-\beta J_{x,y}}$$

for $x, y \in \mathbb{Z}^d$.

(a) Assume that $\sum_{x\in\mathbb{Z}^d} J_{0,x} = +\infty$. Prove that $\mathrm{P}_\beta[0 \longleftrightarrow \infty] = 1$ for all $\beta > 0$, where $\{0 \longleftrightarrow \infty\}$ denotes the event that 0 is connected to Λ_n^c for all $n \geq 1$.

Hint: Use the second lemma of Borel-Cantelli.

From now on, we assume that $\sum_{x\in\mathbb{Z}^d} J_{0,x} < \infty$.

(b) Define the analogues β_c, $\tilde{\beta}_c$, $\phi_\beta(S)$ of p_c, $\tilde{p}_c$, $\phi_p(S)$ in this context.

(c) Show that for all $\beta \geq \tilde{\beta}_c$,

$$\mathrm{P}_\beta[0 \longleftrightarrow \infty] \geq \frac{\beta - \tilde{\beta}_c}{\beta}.$$

Hint: Argue first that for $\beta > 0$ and a finite subset $A \subset \mathbb{Z}^d$,

$$\frac{d}{d\beta}\mathrm{P}_\beta\left[0 \longleftrightarrow A^\mathsf{c}\right] \geq \frac{1}{\beta} \inf_{S\subseteq A, 0\in S} \phi_\beta(S) \cdot \left(1 - \mathrm{P}_\beta\left[0 \longleftrightarrow A^\mathsf{c}\right]\right). \tag{1}$$

Note that a finite volume version of (1) (i.e. with events restricted to the subgraph with vertex set Λ_n) can be obtained analogously to the proof of Lemma 2 in Section 2.2.

(d) Assume that the interactions are finite-range (i.e. $\exists R$ s.t. $J_{x,y} = 0$ if $|x - y| \geq R$). Show that for all $\beta < \tilde{\beta}_c$, there exists $c > 0$ such that

$$\mathrm{P}_\beta[0 \longleftrightarrow \Lambda_n^\mathsf{c}] \leq e^{-cn}.$$

(e) In the general case (i.e. no finite-range assumption), show that for all $\beta < \tilde{\beta}_c$,

$$\sum_{x\in\mathbb{Z}^d} \mathrm{P}_\beta[0 \longleftrightarrow x] < \infty.$$

Deduce that $\tilde{\beta}_c = \beta_c$.

Hint: Consider S with $\phi_\beta(S) < 1$ and show that for all $n \geq 1$,

$$\sum_{x\in\Lambda_n} \mathrm{P}_\beta[0 \overset{\Lambda_n}{\longleftrightarrow} x] \leq \frac{|S|}{1 - \phi_\beta(S)}.$$