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Percolation Theory - Exercise Sheet 5

Exercise 5.1.$^{(\star)}$ **[Properties of the correlation length]**
Show that the correlation length as a function $\xi : [0, p_c] \to [0, \infty]$ is

(a) non-decreasing,

(b) continuous,

(c) satisfies $\xi(0) = 0$ and $\xi(p_c) = \infty$.

Hint: To prove continuity, one can use the following results from analysis:

- A function is continuous if and only if it is both upper- and lower-semicontinuous.
- The pointwise supremum (resp. infimum) of lower (resp. upper) semicontinuous functions is lower (resp. upper) semincontinuous.

To apply these results, argue that

$$\sup_{n\geq 1} -\frac{\log\left(\theta_n(p)\cdot Cn^{d-1}\right)}{n} = \frac{1}{\xi(p)},$$

and similarly express $\frac{1}{\xi(p)}$ as the infimum of continuous functions.

Exercise 5.2.$^{(\star)}$ Let A be an increasing event that depends on finitely many edges only.

(a) Prove that for $p \in (0,1)$ and $\gamma \geq 1$,

$$\mathrm{P}_{p^\gamma}[A] \leq \mathrm{P}_p[A]^\gamma.$$

Hint: Consider a finite set F such that the event A depends only on edges in F. Prove the inequality by induction on $|F|$.

(b) Show that part (a) is equivalent to the following statement: The function

$$\begin{aligned}(0,1) &\longrightarrow [0,+\infty]\\ p &\longmapsto \frac{\log(\mathrm{P}_p[A])}{\log(p)}\end{aligned}$$

is non-increasing.

(c) Use part (b) to prove that the correlation length as a function $\xi : [0, p_c] \to [0, \infty]$ is strictly increasing.

Exercise 5.3. Let $p < p_c$. Prove that there exists a constant $c = c(p,d) > 0$ such that

$$\frac{c \cdot e^{-\frac{\|x\|_1}{\xi(p)}}}{\|x\|_1^{4d(d-1)}} \leq \mathrm{P}_p[0 \longleftrightarrow x] \leq e^{-\frac{\|x\|_\infty}{\xi(p)}}.$$

Exercise 5.4. **[Alternative definition of correlation length via $\phi_p(S)$]**
For $p \in [0,1]$, define

$$\tilde{\xi}(p) := \min\{n \geq 1 : \phi_p(\Lambda_n) \leq 1/e\},$$

where $\min \emptyset = +\infty$ by convention. Prove that there exists a constant $C = C(d) > 0$ such that for all $p \in [0,1]$,

$$\xi(p) \leq \tilde{\xi}(p) \leq 1 + C\xi(p) \log(2 + \xi(p)).$$

Hint: To obtain the first inequality, recall the proof of exponential decay in radius based on $\phi_p(S) < 1$ for some finite, connected set S containing 0. Towards the second inequality, show that $\phi_p(\Lambda_n) \leq 1/e$ for n sufficiently large compared to $\xi(p)$.