Prof. Vincent Tassion
Laurin Köhler-Schindler HS2020

Percolation Theory - Exercise Sheet 6

Recall $p_k = \frac{1}{(\lambda e)^{|\Lambda_k|}}$, $k \geq 1$, *as defined in the context of k-independent site percolation.*

Exercise 6.0. [Exponential decay in volume in the perturbative regime]
Consider Bernoulli *bond* percolation with parameter $p \leq \frac{1}{2d} p_2$. Prove that for all $n \geq 1$,

$$\mathrm{P}_p[|C_0| \geq n] \leq e^{-n}.$$

Remark: The exercise originates from the lecture notes and has later been added to this exercise sheet for completeness.

Exercise 6.1.$^{(\star)}$ **[Alternative def. of correlation length via** $\mathrm{P}_p[\Lambda_n \leftrightarrow \partial\Lambda_{2n}]$**]**
For $p \in [0,1]$, define

$$\ell(p) := \min\{n \geq 1 : \mathrm{P}_p[\Lambda_n \leftrightarrow \partial\Lambda_{2n}]) \leq p_3\}.$$

Prove that there exists a constant $C = C(d) > 0$ such that for all $p \in [0,1]$,

$$\frac{\xi(p)}{2} \leq \ell(p) \leq 1 + C\xi(p)\log(2 + \xi(p)).$$

Hint: To obtain the first inequality, recall the proof of exponential decay in volume via renormalization. Towards the second inequality, show that $\mathrm{P}_p[\Lambda_n \leftrightarrow \partial\Lambda_{2n}]) \leq p_3$ for n sufficiently large compared to $\xi(p)$.

Exercise 6.2. [Volume correlation length]

(a) Let $p \in (0,1)$. Show that for all $m, n \geq 1$,

$$\frac{1}{m+n}\,\mathbb{P}_p\big[|C_0| = m+n\big] \geq \frac{p}{(1-p)^2}\,\frac{1}{n}\,\mathbb{P}_p\big[|C_0| = n\big]\,\frac{1}{m}\,\mathbb{P}_p\big[|C_0| = m\big].$$

(b) Let $p \in [0,1]$. Prove that the volume correlation length, defined by

$$\zeta(p) = \Big(\lim_{n\to\infty} -\frac{1}{n}\log\big(\mathbb{P}_p\big[|C_0| = n\big]\big)\Big)^{-1},$$

is well-defined, and finite for $p < p_c$. Also prove that

$$\mathbb{P}_p\big[|C_0| = n\big] \leq \frac{(1-p)^2}{p}\, n\, e^{-\frac{n}{\zeta(p)}}.$$

(c) Let $p < p_c$. Show that

$$\mathbb{P}_p\big[|C_0| \geq n\big] = e^{-\frac{n}{\zeta(p)} + o(n)}.$$

(d) Prove that there exists a constant $C' = C'(d) > 0$ such that for all $p \in [0,1]$,

$$\xi(p) \leq \zeta(p) \leq 1 + C'\xi(p)^d \log(2 + \xi(p))^d.$$

Hint: Towards the second inequality, first argue that that $\zeta(p) \leq (2\ell(p)+1)^d$.

Exercise 6.3. [Exponential decay in volume]
The goal of this exercise is to give an alternative proof of exponential decay in volume.

(a) Let $X \geq 0$ be a random variable. Assume that $\mathbb{E}[e^{\varepsilon X}] =: C < \infty$ for some $\varepsilon > 0$. Show that for all $x \geq 0$,
$$\mathbb{P}[X \geq x] \leq C\, e^{-\varepsilon x}.$$

Let $p < p_c$. To prove that there exists $c > 0$ such that for all $n \geq 1$
$$\mathbb{P}_p\big[|C_0| \geq n\big] \leq e^{-cn},$$
it therefore suffices to prove $\mathbb{E}[e^{\epsilon |C_0|}] < \infty$ for some $\epsilon > 0$.

(b) Using BK-Reimer inequality, show that
$$\mathbb{E}_p\big[|C_0|^2\big] \leq \mathbb{E}_p\big[|C_0|\big]^3.$$

(c) More generally, show that
$$\mathbb{E}_p\big[|C_0|^n\big] \leq 2^n\, n!\, \mathbb{E}_p\big[|C_0|\big]^{2n-1},$$
and conclude from there.