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Percolation Theory - Exercise Sheet 7

Exercise 7.1.$^{(\star)}$ Let $N(\omega)$ be the number of infinite clusters in the percolation configuration $\omega \in \{0,1\}^E$. Prove that

$$N = \begin{cases} 0 \text{ a.s.} & \text{if } \theta(p) = 0, \\ 1 \text{ a.s.} & \text{if } \theta(p) > 0, \end{cases}$$

where we recall that $\theta(p) = \mathrm{P}_p[0 \longleftrightarrow \infty]$.

Exercise 7.2.$^{(\star)}$

(a) Let $x, y \in \mathbb{Z}^d$. Prove that $p \mapsto \mathrm{P}_p[x \longleftrightarrow y]$ is continuous on $[0,1]$.

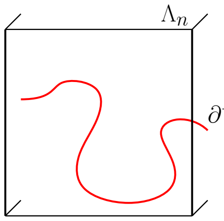
Hint: Using the uniqueness zone, show that the sequence $(f_n)_{n \geq 1}$ defined by $f_n(p) := \mathrm{P}_p[x \overset{\Lambda_n}{\longleftrightarrow} y]$ converges uniformly on $[0,1]$.

(b) Prove that $p \mapsto \theta(p)$ is continuous on $(p_c, 1]$.

Exercise 7.3. Let $p \in [0,1]$ such that $\theta(p) > 0$. Define

$$\partial^- \Lambda_n = \{x \in \Lambda_n : x_1 = -n\}, \; \partial^+ \Lambda_n = \{x \in \Lambda_n : x_1 = n\},$$

which are two opposite sides of the boundary of $\Lambda_n = \{-n, \dots, n\}^d$.
Prove that

$\lim_{n \to \infty} \mathrm{P}_p \Big[\partial^- \Lambda_n$  $\partial^+ \Lambda_n \Big] = 1,$

where the drawing represents an open path in Λ_n from $\partial^- \Lambda_n$ to $\partial^+ \Lambda_n$.

In Chapter 3, uniqueness of the infinite cluster has been proven for Bernoulli percolation on $\mathbb{Z}^d$, $d \geq 2$. More precisely, for $p \in [0,1]$, it holds that either

$$\mathrm{P}_p[N = 0] = 1 \quad \text{or} \quad \mathrm{P}_p[N = 1] = 1.$$

The goal of the following two exercises is to study the number of infinite clusters on more general graphs. To this end, note that Bernoulli percolation naturally extends from $\mathbb{Z}^d$ to general graphs.

Exercise 7.4. **[Infinite clusters on general graphs]** Let $G = (V, E)$ be a connected, locally finite (i.e. every vertex has finite degree) graph.

(a) For any $p \in [0,1]$, prove that $\mathrm{P}_p[N = 0] \in \{0,1\}$ and $\mathrm{P}_p[N = \infty] \in \{0,1\}$. Deduce that $\mathrm{P}_p[1 \leq N < \infty] \in \{0,1\}$.

(b) Give an example of a graph such that for some $p \in [0,1]$ and some $1 \leq k < \ell < \infty$,

$$\mathrm{P}_p[N = k] > 0 \quad \text{and} \quad \mathrm{P}_p[N = \ell] > 0.$$

(c) Let $p \in (0,1)$ and fix an integer $k \geq 1$. Prove that there exists a constant $c > 0$ such that

$$\mathrm{P}_p[N = 1] \geq c \cdot \mathrm{P}_p[N = k].$$

(d) Give an example of a graph such that for some $p \in [0,1]$ and for all $1 \leq k < \infty$,

$$\mathrm{P}_p[N = k] > 0.$$

An important tool in our study of the number of infinite clusters in Bernoulli percolation on $\mathbb{Z}^d$ was translation invariance of the measure P_p, which was used to prove ergodicity. In the next exercise, we extend these ideas to transitive graphs. A *graph automorphism* is a bijection $\varphi : V \to V$ satisfying

$$u \sim v \iff \varphi(u) \sim \varphi(v)$$

for any $u, v \in V$. A graph $G = (V, E)$ is called *transitive* if

$$\forall u, v \in V, \ \exists \text{ graph automorphism } \varphi \text{ such that } \varphi(u) = v.$$

The group of automorphisms $\mathrm{Aut}(G)$ acts

- on V by $\varphi \cdot v = \varphi(v)$,
- on E by $\varphi \cdot \{u, v\} = \{\varphi(u), \varphi(v)\}$,
- on $\{0, 1\}^E$ by $(\varphi \cdot \omega)(e) = \omega(\varphi^{-1} \cdot e)$, and
- on the product-σ-algebra $\mathcal{F}$ by $\varphi \cdot A = \{\varphi \cdot \omega : \omega \in A\}$.

Note that an edge e is open in ω if and only if the edge $\varphi \cdot e$ is open in $\varphi \cdot \omega$. An event $A \in \mathcal{F}$ is called *invariant* if for all $\varphi \in \mathrm{Aut}(G)$,

$$\varphi \cdot A = A.$$

Exercise 7.5. **[Infinite clusters on transitive graphs]** Let $G = (V, E)$ be a transitive, connected, locally finite graph.

(a) Prove $\mathrm{P}_p[A] \in \{0, 1\}$ for any invariant event A. Deduce that $\mathrm{P}_p[N = k] \in \{0, 1\}$ for all $k \in \mathbb{N} \cup \{\infty\}$.

Hint: Verify that the proofs in Section 2.5 (invariance, mixing property, and ergodicity) also apply in the more general setting of transitive graphs.

(b) Using part (b) of Exercise 7.4., prove that there exists $k \in \{0, 1, \infty\}$ such that $\mathrm{P}_p[N = k] = 1$.

(c) Give examples of graphs satisfying

(i) $\mathrm{P}_p[N = \infty] = 1$ for some $p \in [0, 1]$,

(ii) $\mathrm{P}_p[N = \infty] = 0$ for all $p \in [0, 1]$,

(iii) $\mathrm{P}_p[N = \infty] = 1$, $\mathrm{P}_q[N = 1] = 1$ for some $0 < p, q < 1$.