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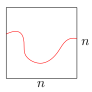
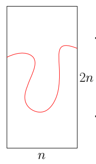
HS2020

Percolation Theory - Exercise Sheet 8

Exercise 8.1.$^{(\star)}$ Consider percolation on $(\mathbb{Z}^2, E)$.

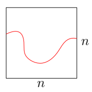
(a) Show that there exists a continuous, strictly increasing $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$ such that for every $p \in [0,1]$ and for all $n \geq 1$,

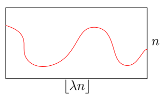
$\mathrm{P}_p\Big[$ 

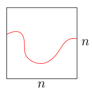
$\Big] \geq g\Big(\mathrm{P}_p\Big[$ 

$\Big]\Big).$

(b) Let $\lambda > 0$. Prove that there exists a continuous, strictly increasing $h_\lambda : [0,1] \to [0,1]$ with $h_\lambda(0) = 0$ and $h_\lambda(1) = 1$ such that for every $p \in [0,1]$ and for all $n \geq 1/\lambda$,

$h_\lambda^{-1}\Big(\mathrm{P}_p\Big[$ 

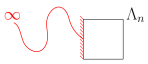
$\Big]\Big) \geq \mathrm{P}_p\Big[$ 

$\Big] \geq h_\lambda\Big(\mathrm{P}_p\Big[$ 

$\Big]\Big)$

Exercise 8.2.$^{(\star)}$ **[Zhang's argument]** Consider percolation on $(\mathbb{Z}^2, E)$.

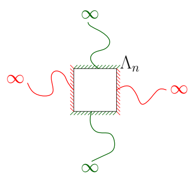
(a) Show that for any $n \geq 1$,

$\mathrm{P}_{1/2}\Big[$ 

$\Big] \geq 1 - \mathrm{P}_{1/2}\left[\Lambda_n \nleftrightarrow \infty\right]^{1/4},$

where the drawing represents an infinite open path from the left side of Λ_n that remains outside of Λ_n.

(b) Show that for any $n \geq 1$,

$\mathrm{P}_{1/2}\Big[$ 

$\Big] \geq 1 - 4\,\mathrm{P}_{1/2}\left[\Lambda_n \nleftrightarrow \infty\right]^{1/4},$

where the drawing represents two infinite primal open paths from the left respectively right side of Λ_n (in red) and two infinite dual open paths from the top respectively bottom side of Λ_n (in green). All paths remain outside of Λ_n and to be precise, the dual open paths actually start at dual vertices that are at distance $1/2$ from the top respectively bottom side.

(c) Using the uniqueness of the infinite cluster, show that $\mathrm{P}_{1/2}\left[\Lambda_n \nleftrightarrow \infty\right] \geq (1/4)^4$ for any $n \geq 1$. Deduce that $\theta(1/2) = 0$.

Exercise 8.3. Let $G = (\mathbb{Z}^d, E)$, $d \geq 2$ and let $k \geq 1$.
Consider a random variable $X = X(e)_{e \in E} \in \{0,1\}^E$ and assume that X is a k-independent (bond) percolation, i.e.

$$\forall A, B \subset E, \quad d(A,B) \geq k \implies \big(X(e)\big)_{e \in A} \text{ and } \big(X(e)\big)_{e \in B} \text{ are independent.}$$

Prove that for some sufficiently small $\delta = \delta(k) > 0$,

$$\Big(\mathrm{P}[X(e) = 1] \geq 1 - \delta,\ \forall e \in E\Big) \implies \mathrm{P}[X \in (0 \longleftrightarrow \infty)] > 0.$$

Hint: Prove it first for $\mathbb{Z}^2$.

Exercise 8.4. Denote by H the subgraph of $(\mathbb{Z}^2, E)$ induced by the set of vertices

$$\{(x,y) \in \mathbb{Z}^2 : 0 \leq y \leq \log(1+x)^2\},$$

and define the critical value of the subgraph H as

$$p_c(H) = \sup\{p \in [0,1] : \mathrm{P}_p[0 \stackrel{H}{\longleftrightarrow} \infty] = 0\}.$$

Prove that $p_c(H) = p_c(\mathbb{Z}^2) = 1/2$.