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HS2020

Percolation Theory - Exercise Sheet 9

Throughout this exercise sheet, we consider percolation on $(\mathbb{Z}^2, E)$.

Exercise 9.1.$^{(\star)}$ **[Quasi-multiplicativity of the 1-arm event]**
Let $p = 1/2$. Define

$$\pi(m, n) := \mathrm{P}_{\frac{1}{2}}[\Lambda_m \longleftrightarrow \partial\Lambda_n]$$

for $m \leq n$. Prove that there exists a constant $c > 0$ such that for all $n_3 \geq n_2 \geq 2n_1$,

$$c\,\pi(n_1, n_2)\pi(n_2, n_3) \leq \pi(n_1, n_3) \leq \pi(n_1, n_2)\pi(n_2, n_3).$$

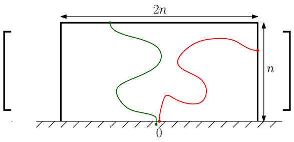
Exercise 9.2.$^{(\star)}$ **[Supercritical correlation length]**
Let $p > 1/2$. Using duality, prove that

$$\xi'(p) = \left(\lim_{n\to\infty} -\frac{\log\left(\mathrm{P}_p[0 \leftrightarrow \partial\Lambda_n, 0 \not\leftrightarrow \infty]\right)}{n}\right)^{-1}$$

is well-defined and show that $\xi'(p) = \frac{1}{2}\xi(1-p)$, where ξ denotes the (subcritical) correlation length (as defined in Chapter 2, Section 3 of the lecture notes).

Exercise 9.3. **[Universal 2-arm exponent in the half-plane]**
Let $p = 1/2$. The goal is to prove the existence of $c, C > 0$ such that for all $n \geq 1$,

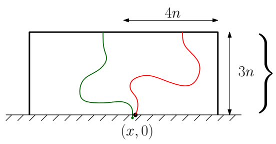
$$c \cdot \frac{1}{n} \leq \mathrm{P}_{\frac{1}{2}}\Bigg[$$



$$\Bigg] \leq C \cdot \frac{1}{n}, \qquad (1)$$

where the drawing denotes the existence on an open path (red) from the origin to $\partial\Lambda_n$ that remains in the upper half-plane and of an open dual path (green) from the dual vertex $(-\frac{1}{2}, -\frac{1}{2})$ to a dual vertex next to $\partial\Lambda_n$ that remains in the upper half-plane and whose first edge is $\{(-\frac{1}{2}, -\frac{1}{2}), -\frac{1}{2}, \frac{1}{2})\}$. For $x \in \mathbb{Z}$, define the event

$$A_x := \Bigg\{$$



$$\Bigg\},$$

where in the drawing the open path (red) starts at $(x, 0)$ and the dual open path (green) starts at $(x - \frac{1}{2}, -\frac{1}{2})$.

(a) Show that there exists a constant $c' > 0$ such that for all $n \geq 1$,

$$\mathrm{P}_{\frac{1}{2}}[A_0] \geq c' \cdot \frac{1}{n},$$

and deduce the lower bound in (1).

Denote by N the (random) number of disjoint open paths from $[-n, n] \times \{0\}$ to $\partial\Lambda_{3n}$ in the upper half-plane.

(b) Show that $\sum_{x \in [-n,n]} \mathbb{1}_{A_x} \le N$.

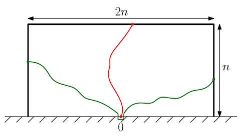
(c) Using the BK-Reimer inequality, show that $\mathbb{E}_{\frac{1}{2}}[N] \le C'$ for a constant C' that does not depend on n.

(d) Combining (b) and (c), prove that there exists a constant $C'' > 0$ such that for all $n \ge 1$,

$$\mathrm{P}_{\frac{1}{2}}[A_0] \le C'' \cdot \frac{1}{n},$$

and deduce the upper bound in (1).

Exercise 9.4. [Universal 3-arm exponent in the half-plane]
Let $p = 1/2$. Prove that there exists $c, C > 0$ such that for all $n \ge 1$,

$$c \cdot \frac{1}{n^2} \le \mathrm{P}_{\frac{1}{2}}\Bigg[$$


$$\Bigg] \le C \cdot \frac{1}{n^2},$$

where the drawing denotes the existence on an open path (red) from the origin to $\partial\Lambda_n$, an open dual path (green, on the left of the red path) from the dual vertex $(-\frac{1}{2}, -\frac{1}{2})$ to a dual vertex next to $\partial\Lambda_n$, and an open dual path (green, on the right of the red path) from the dual vertex $(\frac{1}{2}, -\frac{1}{2})$ to a dual vertex next to $\partial\Lambda_n$. All paths remain in the upper half-plane.
Hint: Try to implement a similar strategy as in Exercise 9.3.