Percolation theory

ETH Zürich, Fall 2020

Organization

Coordinator: Laurin Köhler-Schindler (laurin.koehler-schindler@math.ethz.ch)

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Website: https://metaphor.ethz.ch/x/2020/hs/401-4607-59L/

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Exercises: Weekly on the website. Exercises with a star (*) can be handed in. In class or by email to L. Köhler-Schindler.

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Exercise Classes: October 13, November 10, December 8.

Percolation: applied motivations

Percolation: applied motivations

How does water flow in rocks?

Percolation: applied motivations

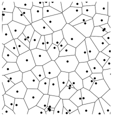
How does water flow in rocks?

How do fires propagate in forests?

Interactions with other fields

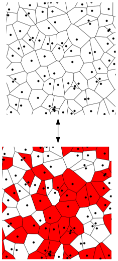
Interactions with other fields

Stochastic geometry



Interactions with other fields

Stochastic geometry



Voronoi percolation

[Vahidi-Asl Wierman 90]

Interactions with other fields

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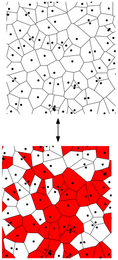
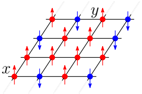


Voronoi percolation

[Vahidi-Asl Wierman 90]

Spin systems



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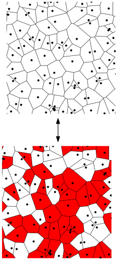

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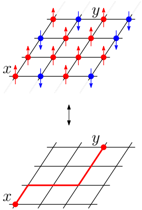
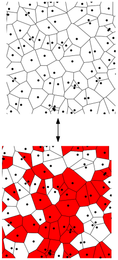


FK percolation

[Fortuin Kasteleyn 74]

Interactions with other fields

Stochastic geometry



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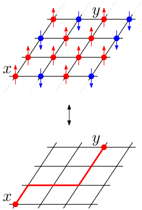
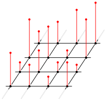


FK percolation

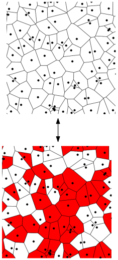
[Fortuin Kasteleyn 74]

Random functions



Interactions with other fields

Stochastic geometry



Voronoi percolation

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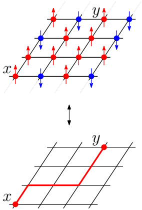

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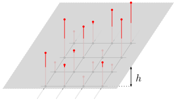

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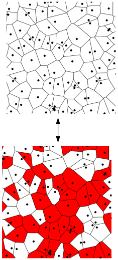

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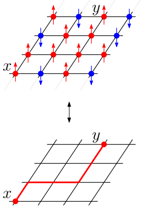

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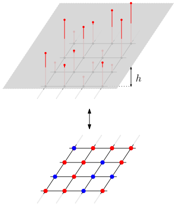

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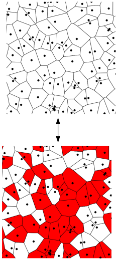

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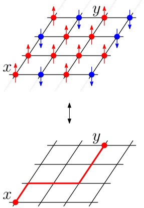

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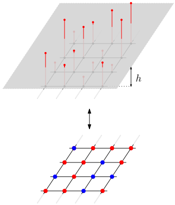
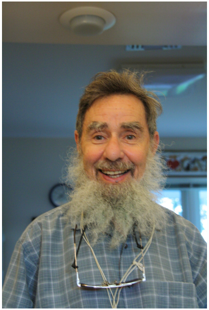


GFF percolation
Nodal lines

Percolation: main motivation!

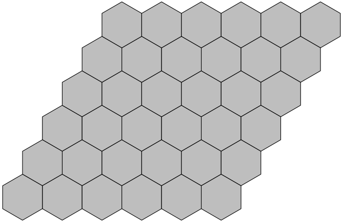


Quite apart from the fact that percolation theory had its origin in an honest applied problem (see Hammersley and Welsh (1980)), it is a source of fascinating problems of the best kind a mathematician can wish for: problems which are easy to state with a minimum of preparation, but whose solutions are (apparently) difficult and require new methods.

Harry Kesten
Percolation theory for mathematicians,
July 1982.

Bernoulli site percolation [Broadbent and Hammersley, 1957]

We tile a lozenge with hexagons.



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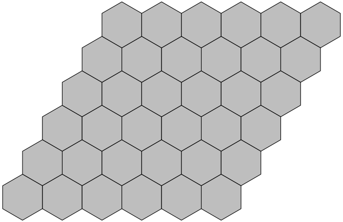

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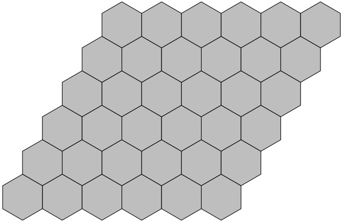

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Random coloring of the hexagons:

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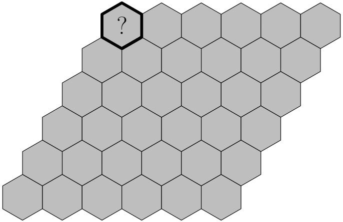

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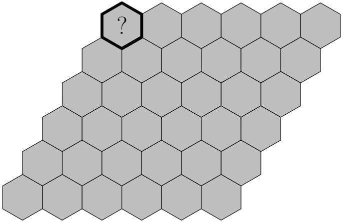

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Random coloring of the hexagons:

A given hexagon is colored:

- red with probability p,

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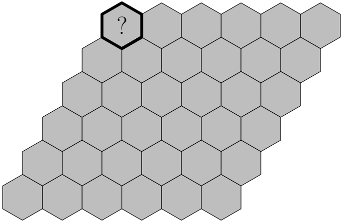

Parameter: $\boxed{0 \leqslant p \leqslant 1}$.

Random coloring of the hexagons:

A given hexagon is colored:

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- blue with probability $1 - p$.

Bernoulli site percolation [Broadbent and Hammersley, 1957]

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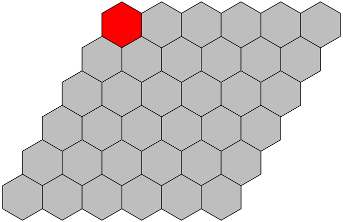

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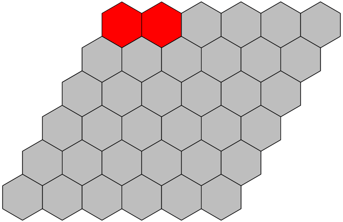

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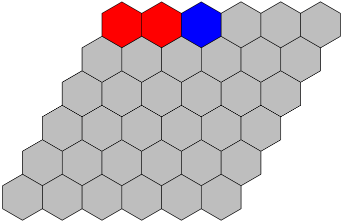

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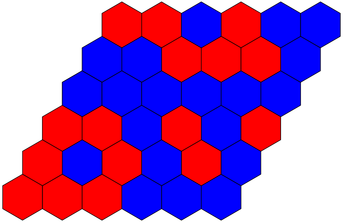

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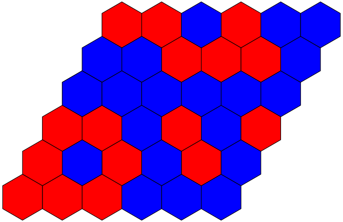

$p = \frac{1}{2}$

Parameter: $\boxed{0 \leqslant p \leqslant 1}$.

Random coloring of the hexagons:

A given hexagon is colored:

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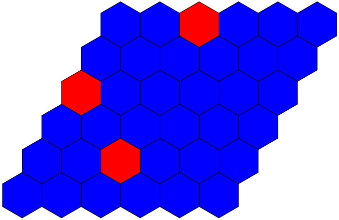
- blue with probability $1 - p$.

Bernoulli site percolation [Broadbent and Hammersley, 1957]

We tile a lozenge with hexagons.



$p = \frac{1}{10}$

Parameter: $\boxed{0 \leqslant p \leqslant 1}$.

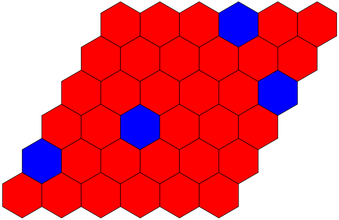
Random coloring of the hexagons:

A given hexagon is colored:

- red with probability p,
- blue with probability $1 - p$.

Bernoulli site percolation [Broadbent and Hammersley, 1957]

We tile a lozenge with hexagons.



$p = \frac{9}{10}$

Parameter: $\boxed{0 \leqslant p \leqslant 1}$.

Random coloring of the hexagons:

A given hexagon is colored:

- red with probability p,
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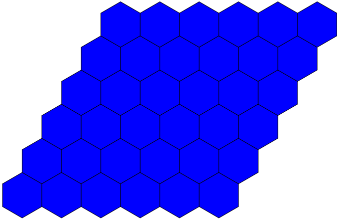
Bernoulli site percolation [Broadbent and Hammersley, 1957]

We tile a lozenge with hexagons.



$p = 0$

Parameter: $\boxed{0 \leqslant p \leqslant 1}$.

Random coloring of the hexagons:

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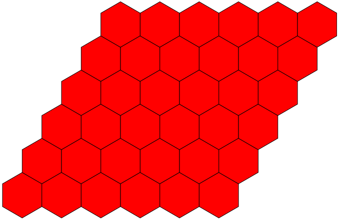
Bernoulli site percolation [Broadbent and Hammersley, 1957]

We tile a lozenge with hexagons.



$p = 1$

Parameter: $\boxed{0 \leqslant p \leqslant 1}$.

Random coloring of the hexagons:

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Bernoulli site percolation [Broadbent and Hammersley, 1957]

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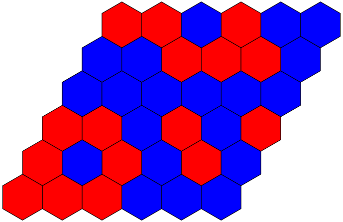

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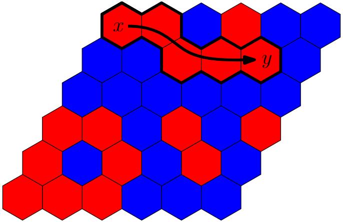


Parameter: $\boxed{0 \leqslant p \leqslant 1}$.

Random coloring of the hexagons:

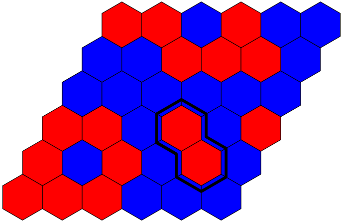
A given hexagon is colored:

- red with probability p,
- blue with probability $1 - p$.

Red path: a path made of red hexagons.

Bernoulli site percolation [Broadbent and Hammersley, 1957]

We tile a lozenge with hexagons.



Parameter: $\boxed{0 \leqslant p \leqslant 1}$.

Random coloring of the hexagons:

A given hexagon is colored:

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Red path: a path made of red hexagons.

Red Cluster: red connected component.
"Island"

Bernoulli site percolation [Broadbent and Hammersley, 1957]

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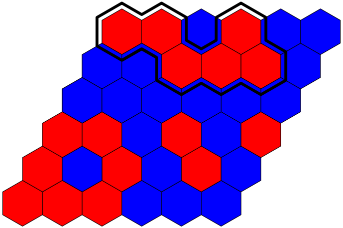

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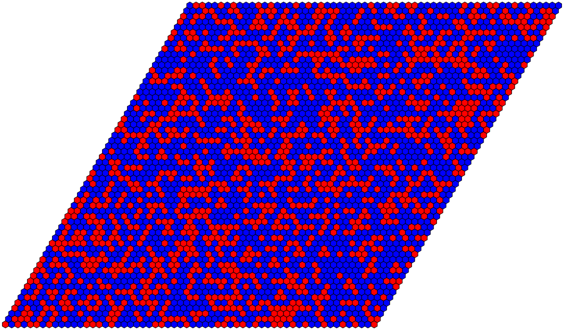
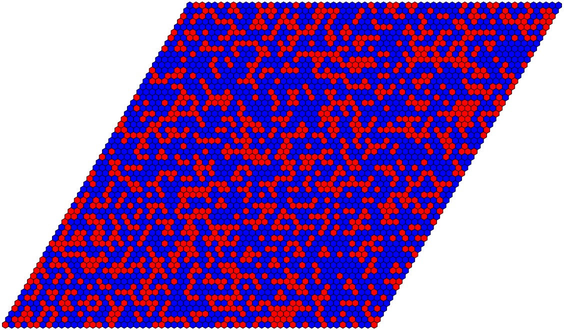
A porous stone?

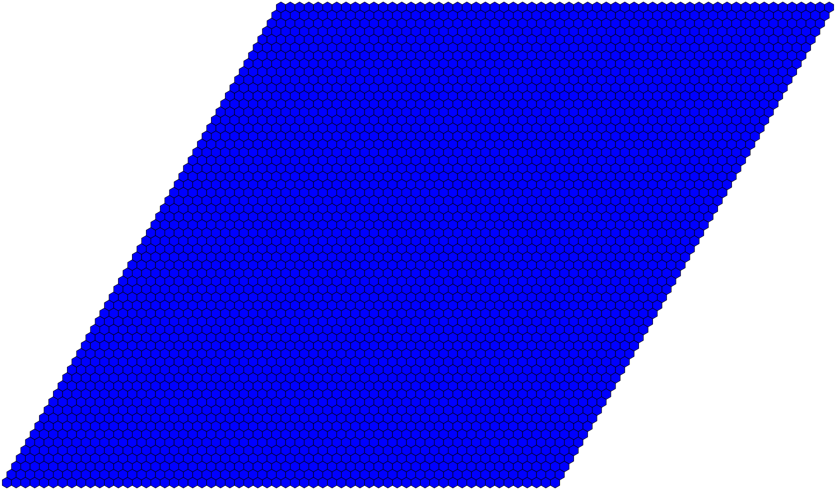


QUESTION 1:

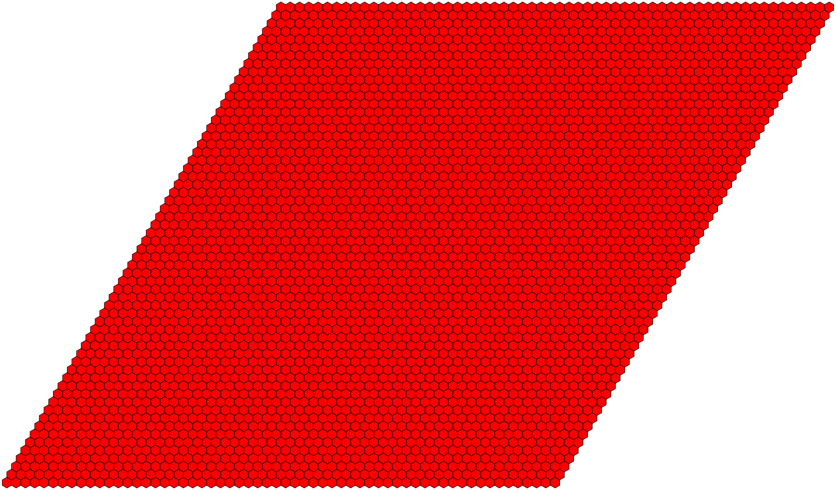


Is there a red path from top to bottom in a large lozenge?

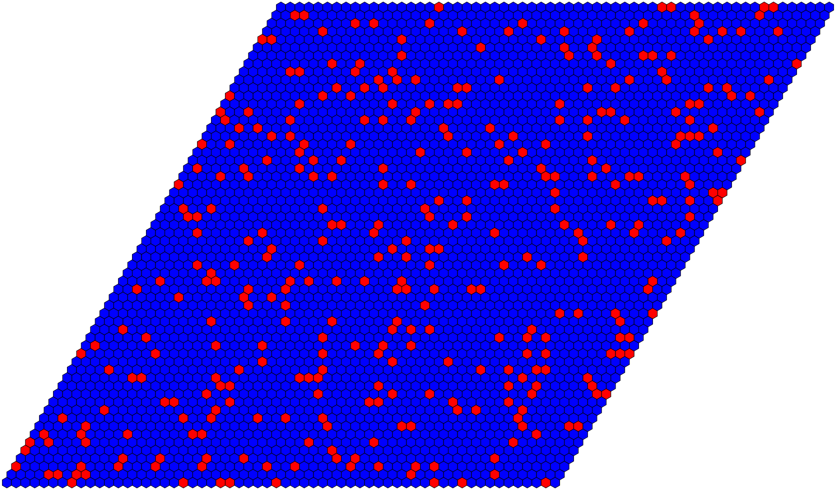
$p = 0$



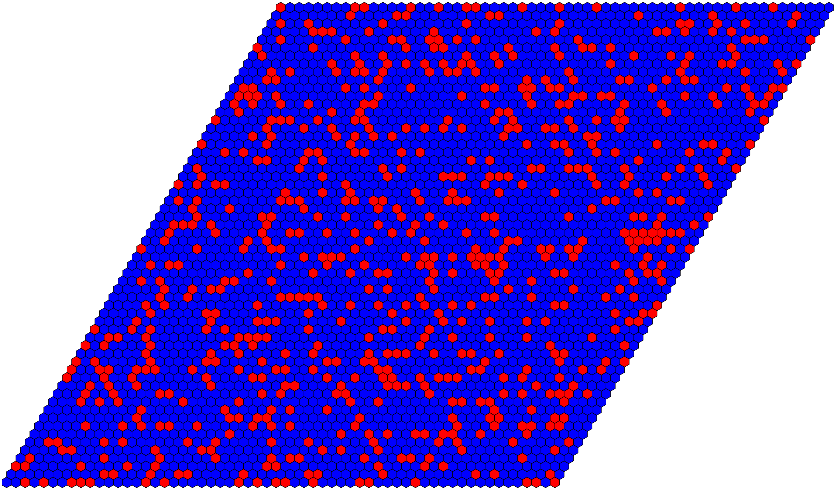
$p = 1$



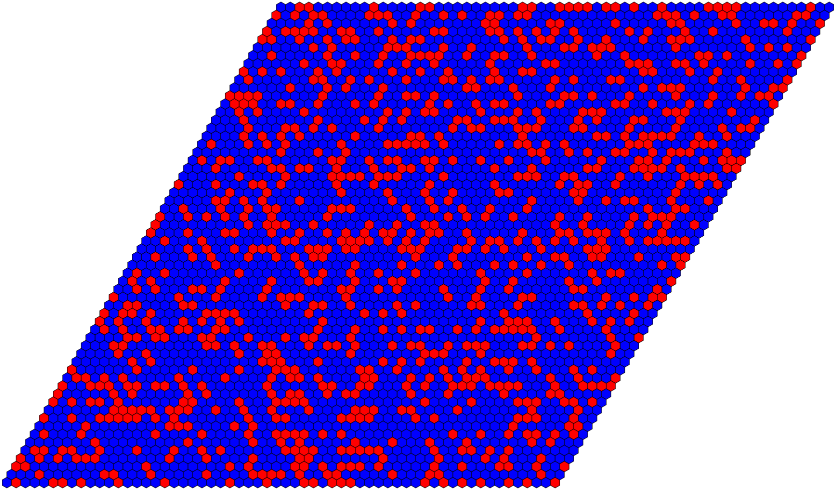
$p = 0.1$



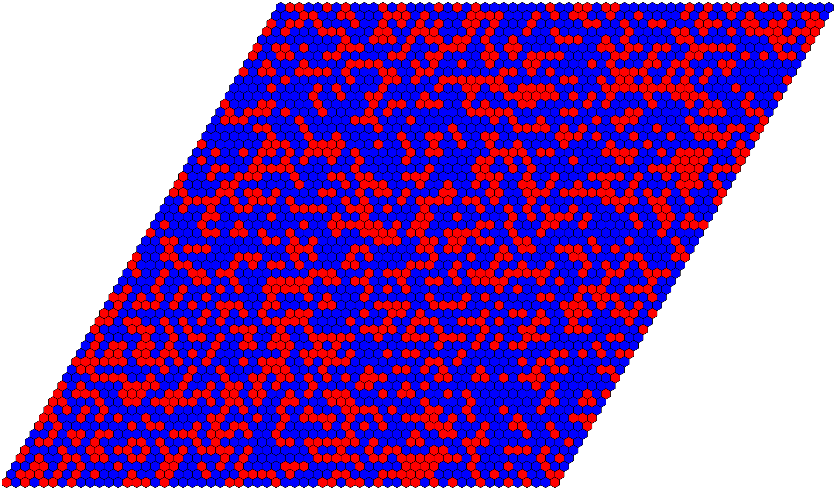
$p = 0.2$



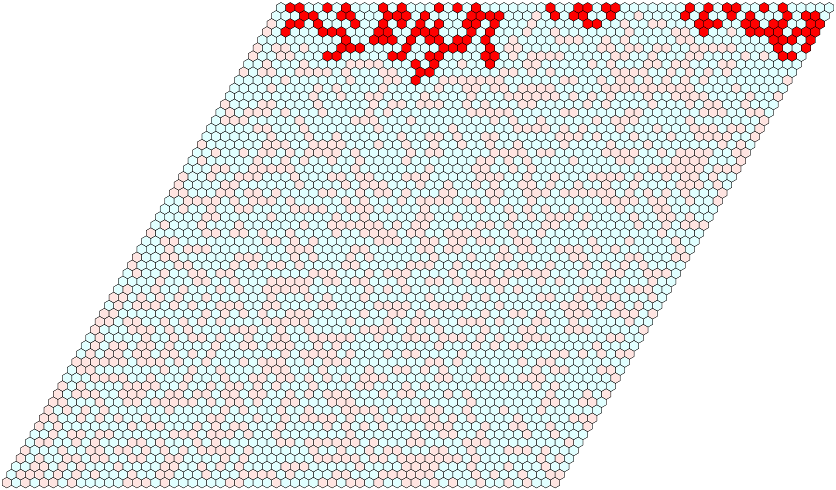
$p = 0.3$



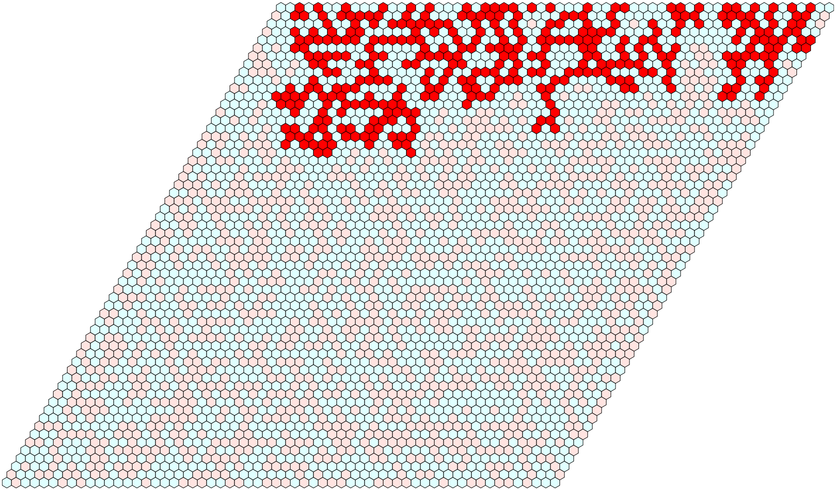
$p = 0.4$



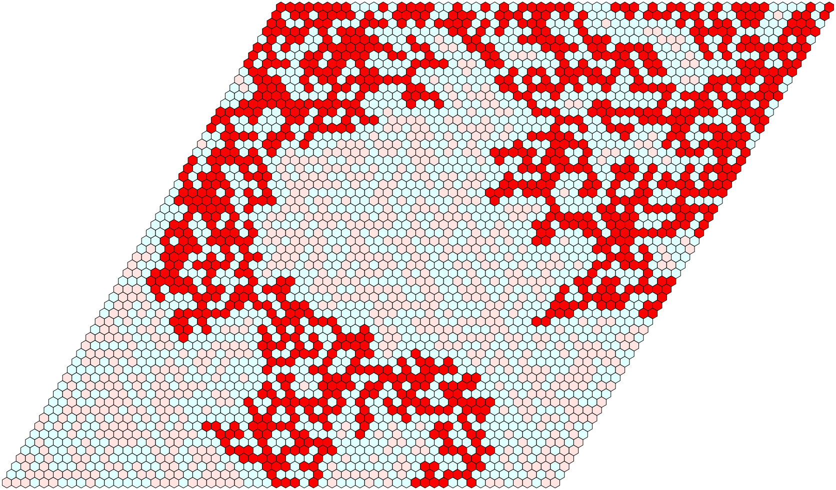
$p = 0.4$



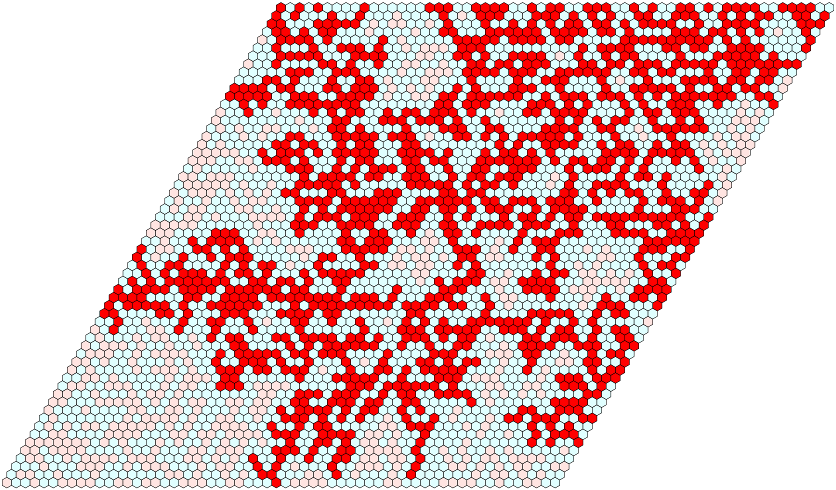
$p = 0.45$



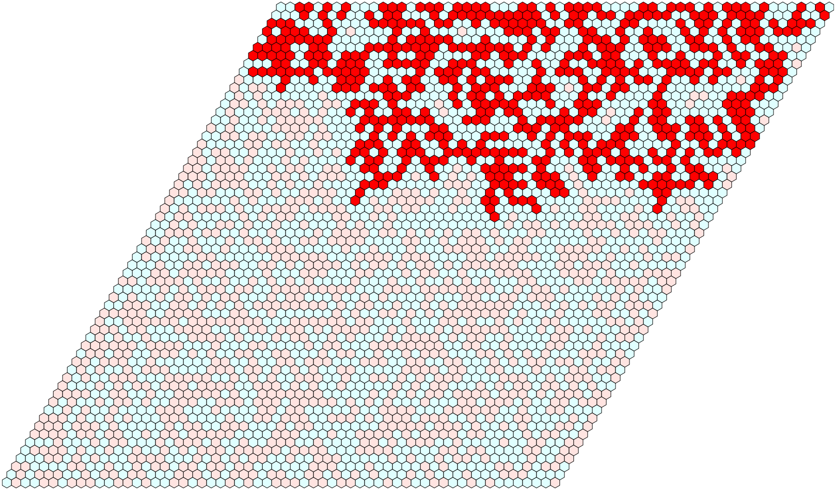
$p = 0.5$



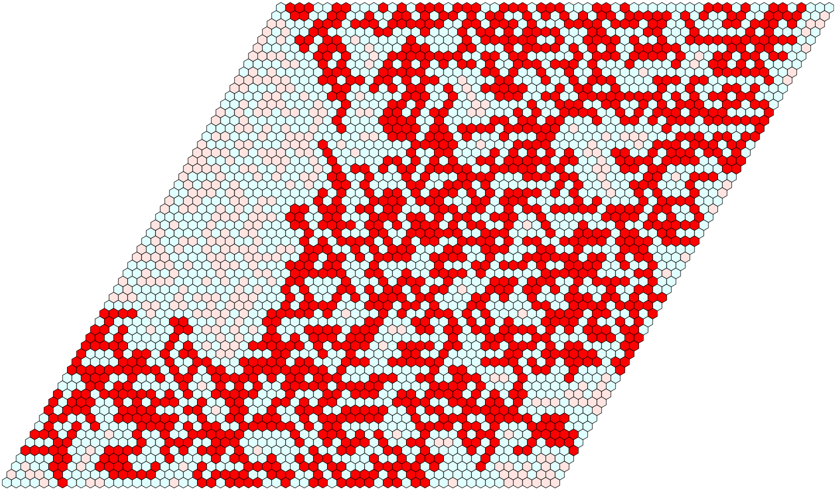
$p = 0.5$



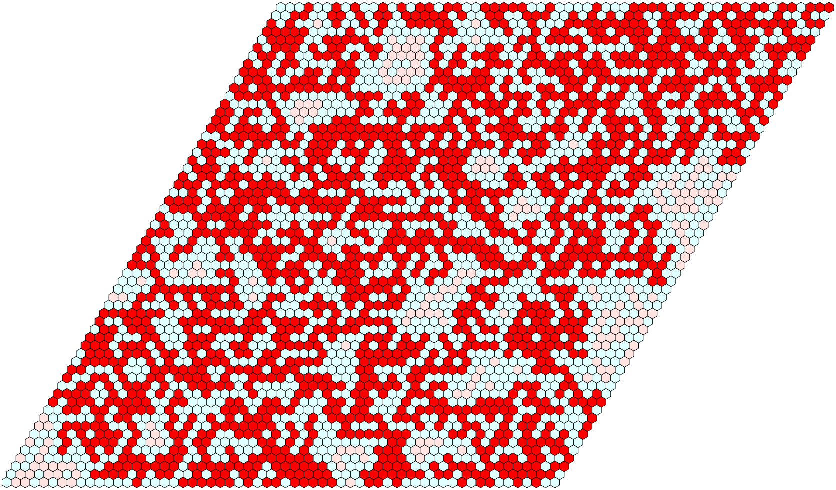
$p = 0.5$



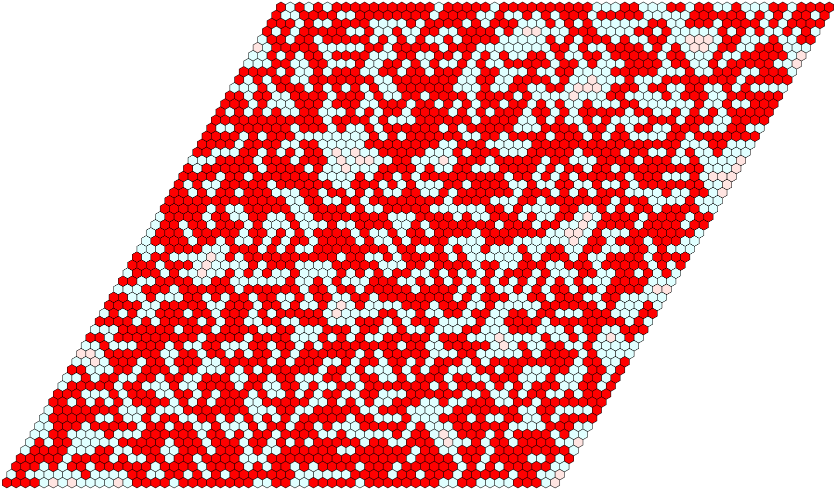
$p = 0.5$



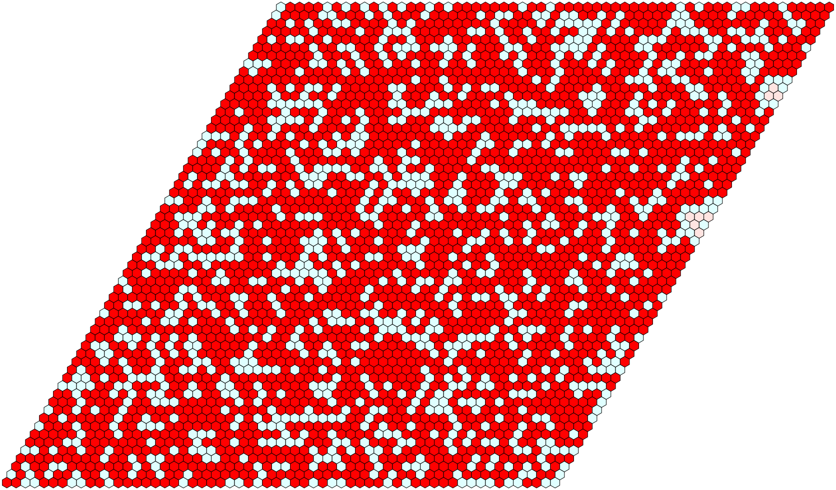
$p = 0.55$



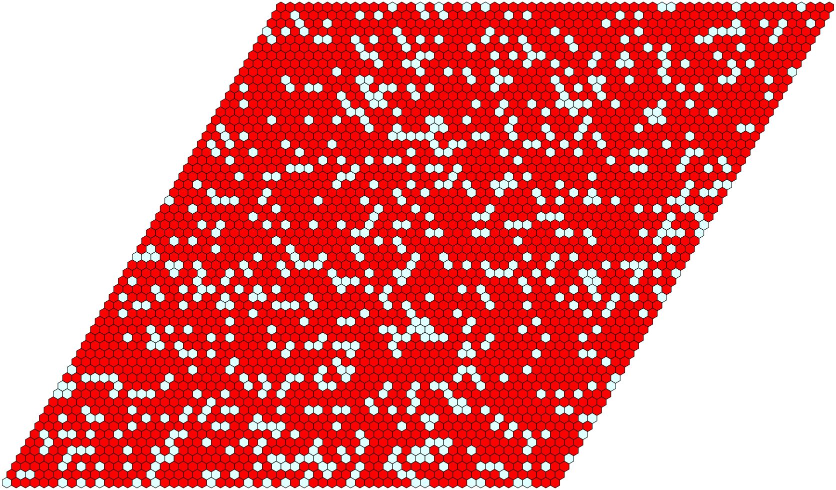
$p = 0.6$



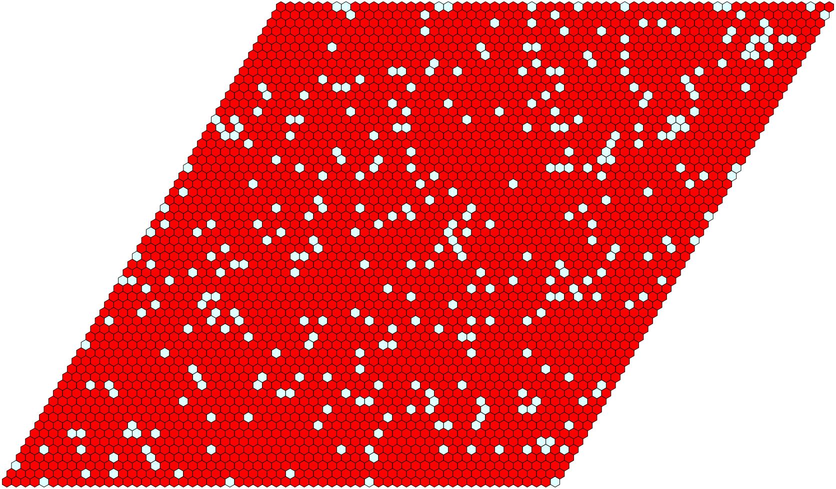
$p = 0.7$



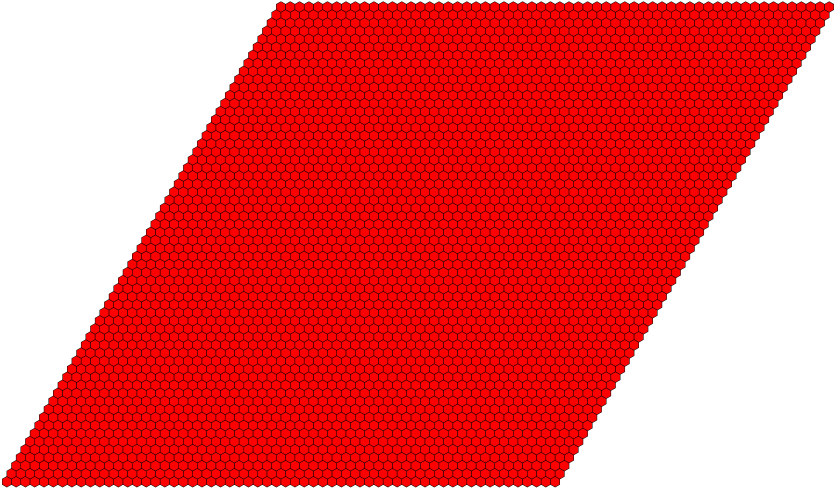
$p = 0.8$

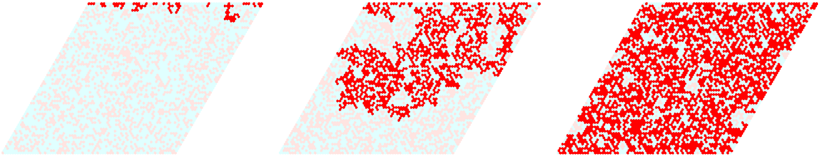


$p = 0.9$

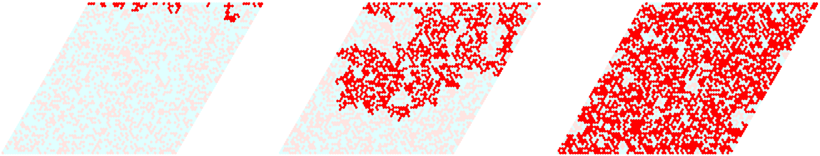


$p = 1$





$p < \frac{1}{2}$ $\quad$ $p = \frac{1}{2}$ $\quad$ $p > \frac{1}{2}$



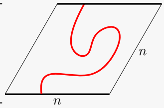
$p < \frac{1}{2}$ $\qquad$ $p = \frac{1}{2}$ $\qquad$ $p > \frac{1}{2}$

Rigorous answer to Question 1

Theorem [Kesten, 1980]

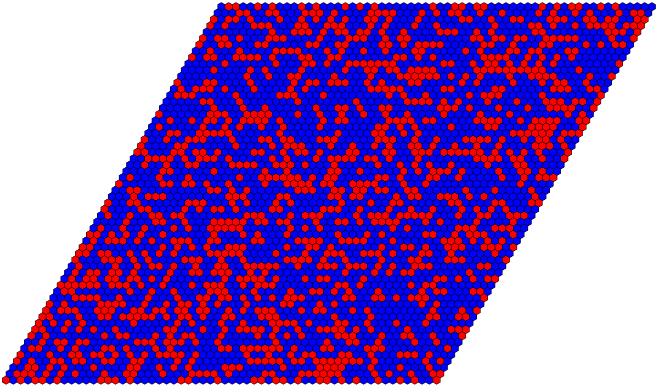
For percolation with parameter p, we have

$$\lim_{n\to\infty} \mathbf{Prob}_p \left[\; \cdot \; \right]$$

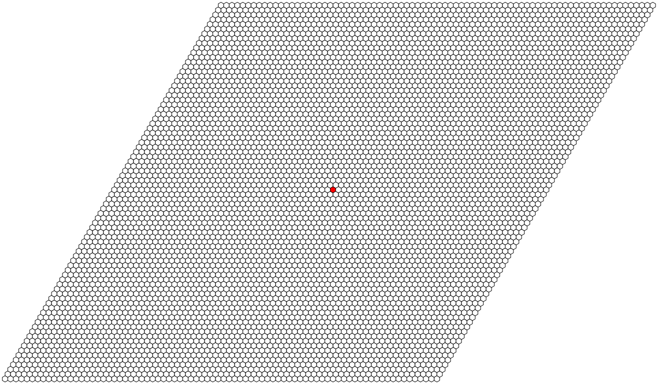


$$= \begin{cases} 0 & \text{if } p < \frac{1}{2} \\ \frac{1}{2} & \text{if } p = \frac{1}{2} \\ 1 & \text{if } p > \frac{1}{2} \end{cases}$$

A forest?

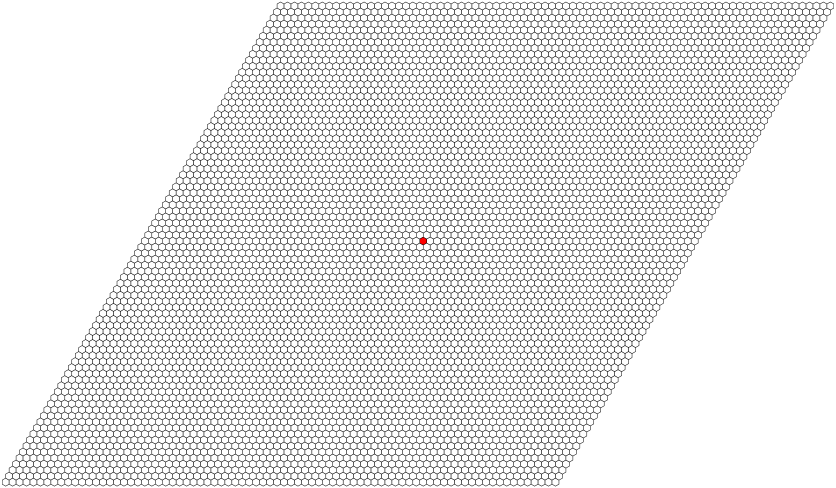


QUESTION 2:

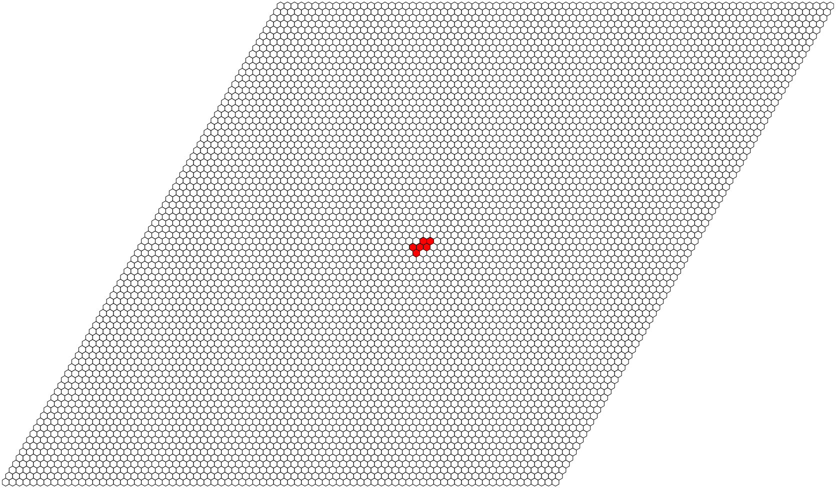


How far can we go when starting from a single hexagon in the center?

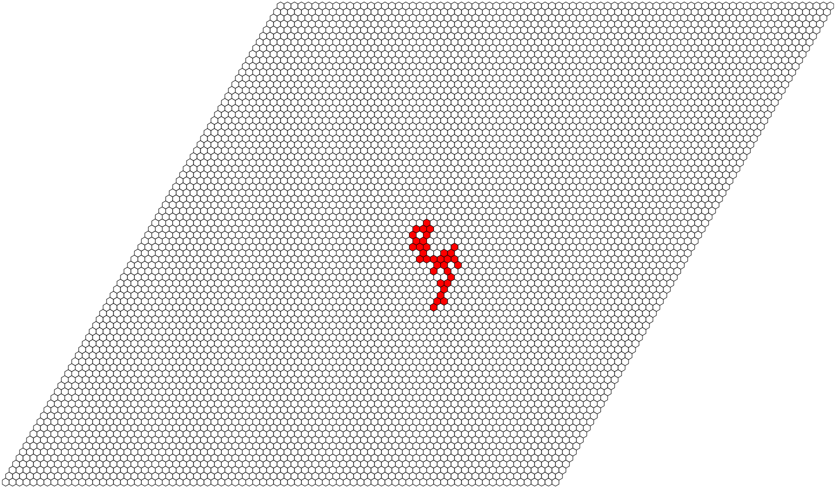
$p = 0$



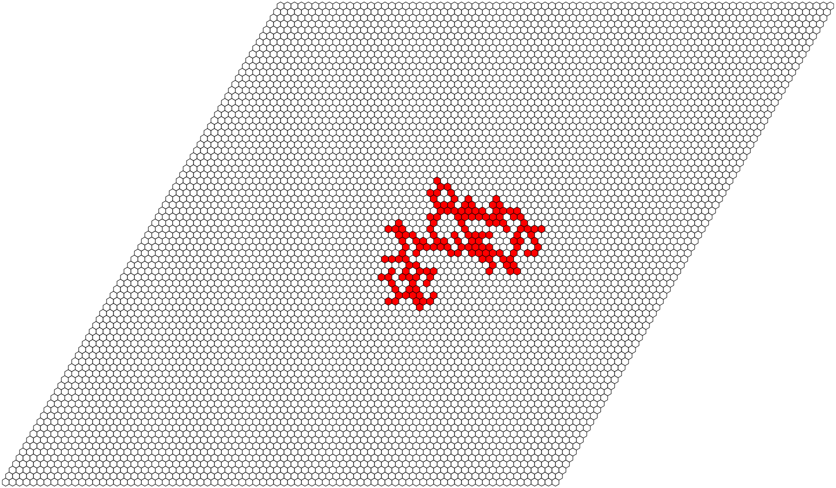
$p = 0.3$



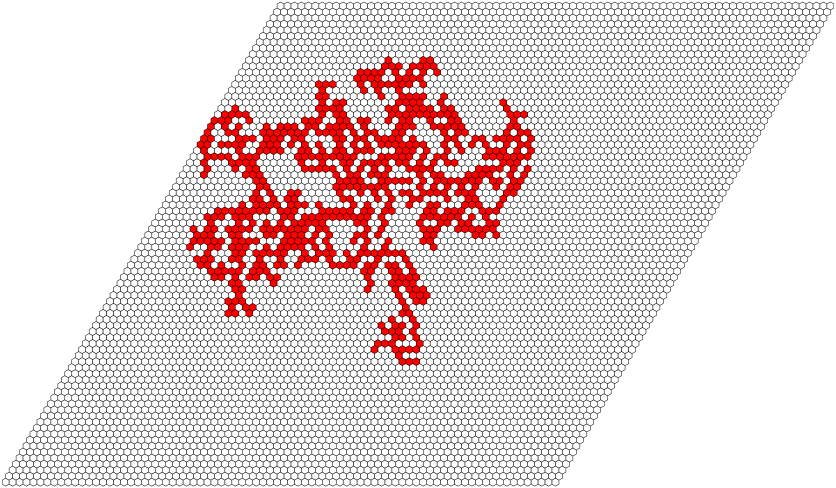
$p = 0.4$



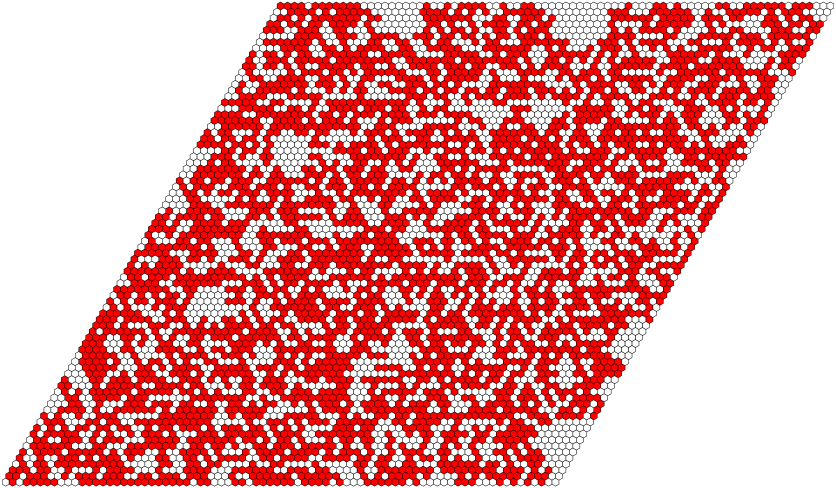
$p = 0.45$



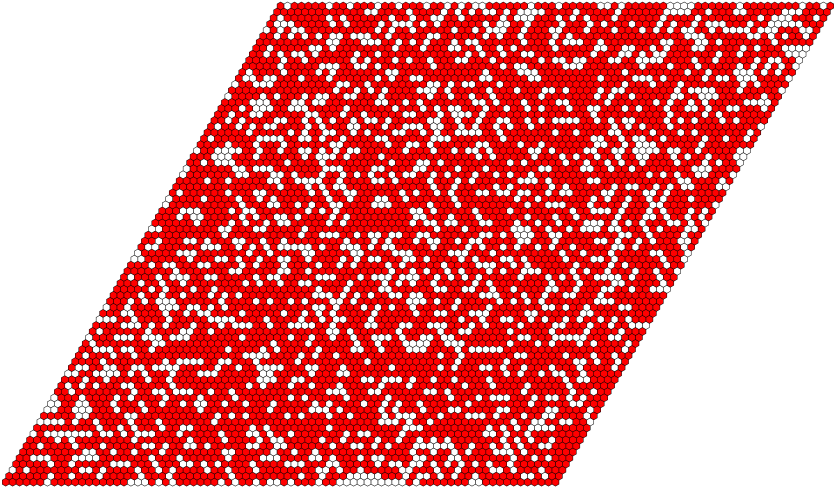
$p = 0.5$



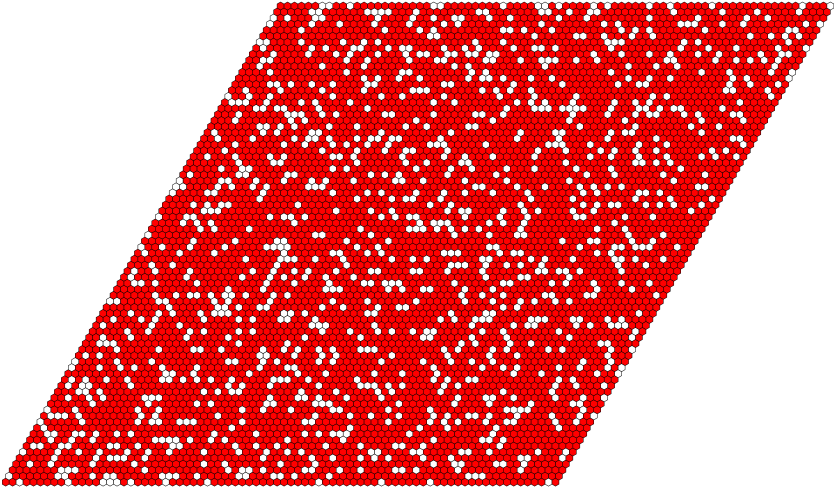
$p = 0.6$



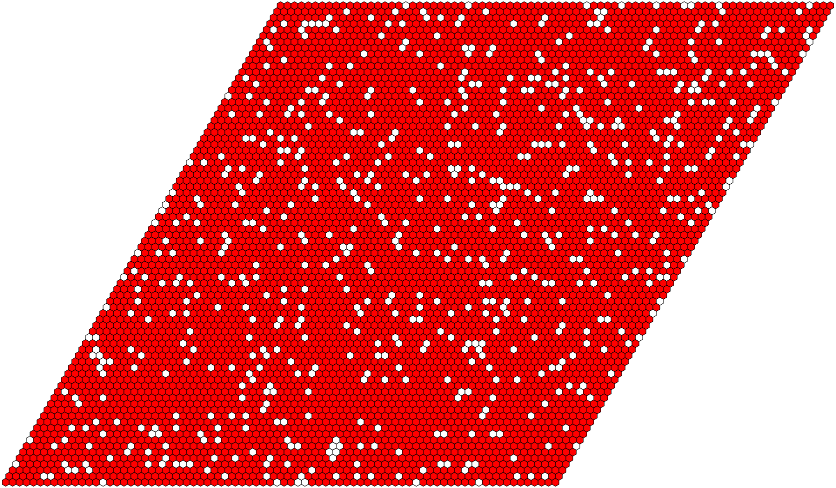
$p = 0.7$



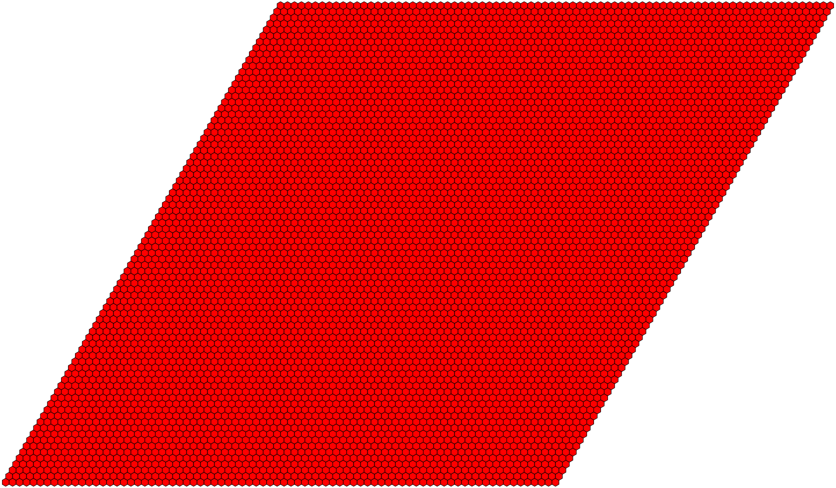
$p = 0.8$

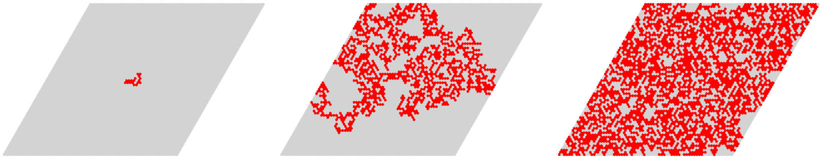


$p = 0.9$

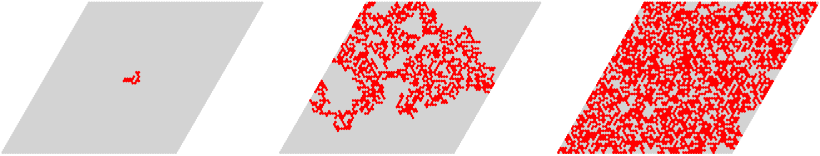


$p = 1$





$p < \frac{1}{2}$ $p = \frac{1}{2}$ $p > \frac{1}{2}$



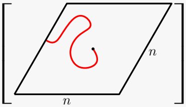
$p < \frac{1}{2}$ $\qquad$ $p = \frac{1}{2}$ $\qquad$ $p > \frac{1}{2}$

Rigorous answer to Question 2

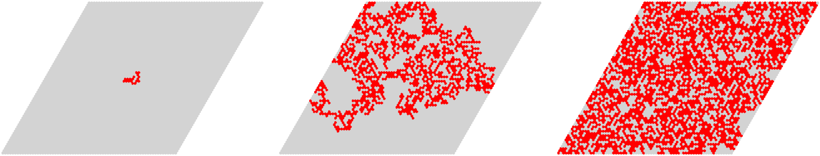
Theorem [Kesten, 1980]

For percolation with parameter p, we have

$\mathbf{Prob}_p$



$$\begin{cases} \leqslant e^{-c(p)n} & \text{if } p < \frac{1}{2}, \quad \textbf{[exponential decay]} \\ \leqslant \frac{1}{n^{c(p)}} & \text{if } p = \frac{1}{2}, \quad \textbf{[polynomial decay]} \\ \geqslant c(p) & \text{if } p > \frac{1}{2}. \quad \textbf{[uniform positivity]} \end{cases}$$

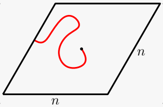


$p < \frac{1}{2}$ $\qquad$ $p = \frac{1}{2}$ $\qquad$ $p > \frac{1}{2}$

RIGOROUS ANSWER TO QUESTION 2

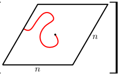
Theorem [Kesten, 1980]

For percolation with parameter p, we have

$\mathbf{Prob}_p$ [

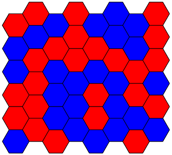
]

$$\begin{cases} \leqslant e^{-c(p)n} & \text{if } p < \frac{1}{2}, \quad \textbf{[exponential decay]} \\ \leqslant \frac{1}{n^{c(p)}} & \text{if } p = \frac{1}{2}, \quad \textbf{[polynomial decay]} \\ \geqslant c(p) & \text{if } p > \frac{1}{2}. \quad \textbf{[uniform positivity]} \end{cases}$$

Remark: For $p = \frac{1}{2}$, $\mathbf{Prob}_p$ [

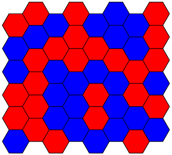
] $\simeq \frac{1}{n^{5/48}}$ [Lawler, Schramm, Werner '02]

Some percolation processes:

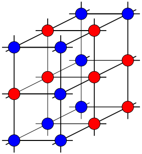


Percolation
on hexagons.

Some percolation processes:

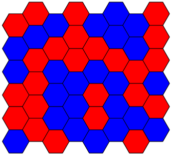


Percolation
on hexagons.

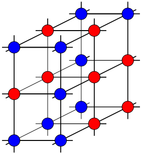


Percolation
on $\mathbb{Z}^d$, $d \geqslant 2$.

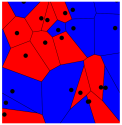
Some percolation processes:



Percolation on hexagons.

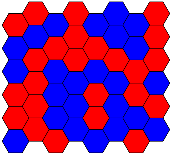


Percolation on $\mathbb{Z}^d$, $d \geqslant 2$.

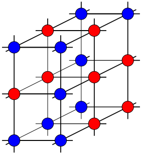


Voronoi percolation in $\mathbb{R}^d$.

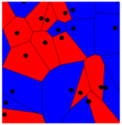
Some percolation processes:



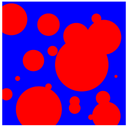
Percolation on hexagons.



Percolation on $\mathbb{Z}^d$, $d \geqslant 2$.

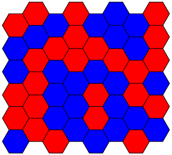


Voronoi percolation in $\mathbb{R}^d$.

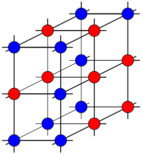


Boolean percolation in $\mathbb{R}^d$.

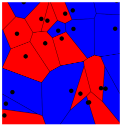
Some percolation processes:



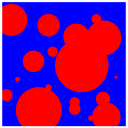
Percolation
on hexagons.



Percolation
on $\mathbb{Z}^d$, $d \geqslant 2$.

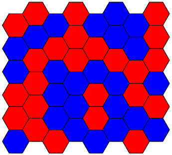


Voronoi percolation
in $\mathbb{R}^d$.

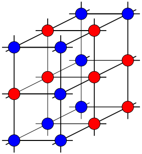


Boolean percolation
in $\mathbb{R}^d$.

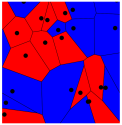
Some percolation processes:



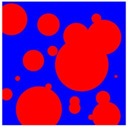
Percolation on hexagons.



Percolation on $\mathbb{Z}^d$, $d \geqslant 2$.

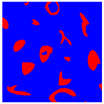


Voronoi percolation in $\mathbb{R}^d$.

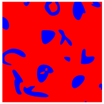


Boolean percolation in $\mathbb{R}^d$.

Phase transition ($p =$ density of red points).



All the red clusters are bounded.



One giant red cluster.

p_c: **critical parameter** (depends on the model).