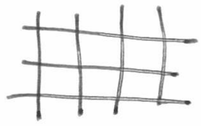
INTRODUCTION

I BERNOULLI BOND PERCOLATION ON $\mathbb{Z}^d$

Setup: • $G = (\mathbb{Z}^d, E)$ $\quad E = \{\{x,y\} \in \mathbb{Z}^d \ \|x-y\|_1 = 1\}$

$d=1$

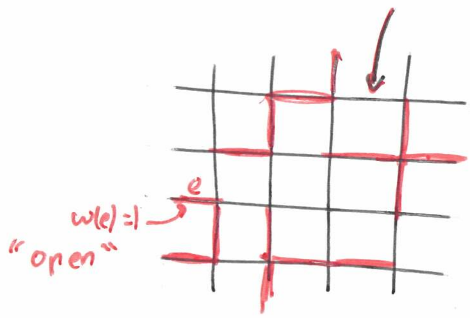


$d=2$

$d=3$ ...

• $p \in [0,1]$

$\omega(e) = 0$ "closed"



edge is open with proba p.

• closed with proba. $(1-p)$

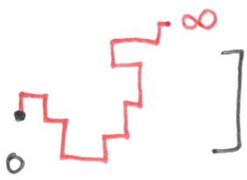
$\hookrightarrow$ percolation configuration $\omega = (\omega(e))_{e \in E} \in \{0,1\}^E$

Probabilistic framework $(\Omega, \mathcal{F}, \mathbb{P}_p)$

$\Omega = \{0,1\}^E$ $\qquad \mathcal{F}$ product σ-algebra

$\mathbb{P}_p = (\text{Bernoulli}(p))^{\otimes E}$ $\qquad$ product measure.

percolation probability

$\theta(p)=\theta_d(p)=P_p\Big[$  $\Big]$

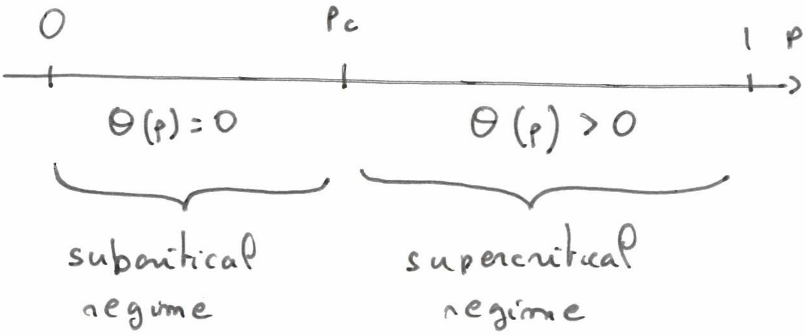
$0 \leftrightarrow \infty$

$0\leftrightarrow\infty$ is a well-defined event (see exercise sheet 1)

Prop: $\boxed{\theta : [0,1] \longrightarrow [0,1] \text{ is non decreasing}}$

Proof: see chapter 1.

Def: $\boxed{p_c = p_c(d) = \sup\{p \in [0,1] : \theta(p)=0\}}$



$\underline{d=1}$

$\forall n \geq 1 \quad \theta(p) = P_p[0 \leftrightarrow \infty] \leq P_p[\exists \text{ path of length } n \text{ from } 0]$

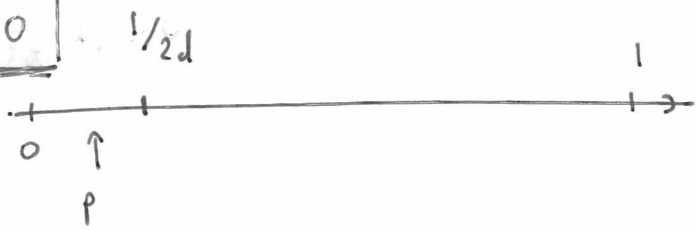
$$\leq 2p^n$$

Hence $\theta(p)=0$ if $p<1 \rightarrow p_c = 1$

Theorem: If $d \geq 2$

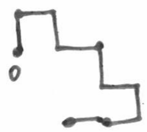
$$\boxed{0 < p_c(d) < 1}$$

Proof $\boxed{p_c > 0}$.



goal: prove that $\forall p < \frac{1}{2d}$ $\quad \theta(p) = 0$

Let $\Pi_n = \{ \pi$ edge self-avoiding paths length n from $0 \}$



$\pi \in \Pi_9$

Let $n \geq 1$

$$\theta(p) \leq \mathbb{P}_p \Big[\bigcup_{\pi \in \Pi_n} \{\text{all the edges of } \pi \text{ are open}\} \Big]$$

$$\leq \sum_{\pi \in \Pi_n} \underbrace{\mathbb{P}_p [\text{all the edges of } \pi \text{ are open}]}_{p^n}$$

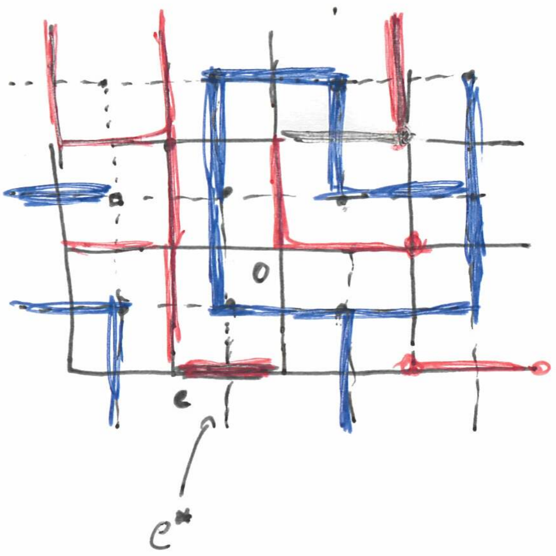
$$= |\Pi_n| \, p^n$$

$$\leq 2d p^n \xrightarrow[\text{if } p < \frac{1}{2d}]{} 0$$

$p_c(2) < 1$

Duality:

$G^* = ((\mathbb{Z}^2)^*, E^*)$ copy of $(\mathbb{Z}^2, E)$ translated by $(\frac{1}{2}, \frac{1}{2})$



bijection: $E \longrightarrow E^*$

$e \longmapsto e^*$

$\Omega \longrightarrow \Omega^*$

$\omega \longmapsto \omega^*$ $\qquad \omega^*(e^*) = 1 - \omega(e)$

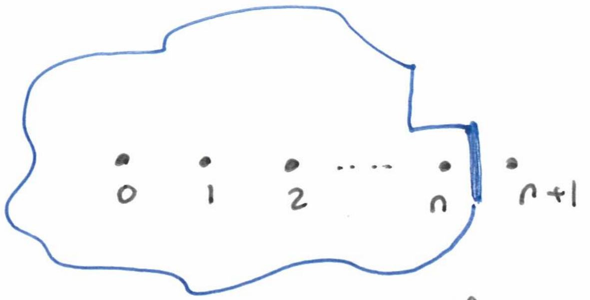
$\omega \sim P_p \longrightarrow \omega^* \sim P_{1-p}$

Key result: $\quad$ o is not connected to ∞ in ω

iff o is surrounded by a circuit in ω^*

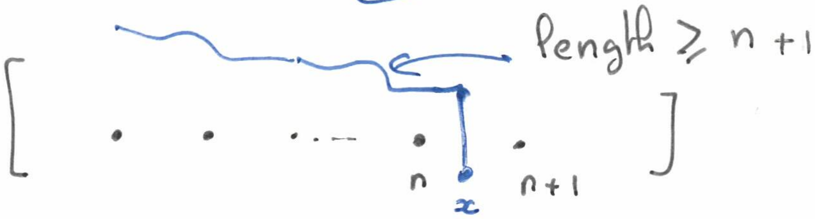
$$1-\theta(p) = P_p\left[\, 0 \not\leftrightarrow \infty \,\right]$$

$$= P_p\left[\, \exists n \text{ s.t. } \right.$$



$$\left. \right]$$

$$\leq \sum_{n \geq 0} \underbrace{P_p\left[\quad \right]}_{\leq \left(4(1-p)\right)^{n+1}}$$



$$= \sum_{n \geq 0} \left(4(1-p)\right)^{n+1} \overset{p > \frac{3}{4}}{=} \frac{4(1-p)}{1-\left(4(1-p)\right)} \overset{p > \frac{7}{8}}{<} 1$$

CCL: $\theta(p) > 0$ if $p > \frac{7}{8}$

BERNOULLI PERCOLATION ON $\mathbb{Z}^d$

I DEFINITIONS

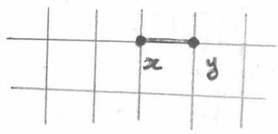
1.1. GRAPH TERMINOLOGY.

For $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, $\|x\|_1 := \sum_{i=1}^{d} |x_i|$ (L^1 norm)

Graph structure on $\mathbb{Z}^d$:

$$E = \{ \{x, y\} \subset \mathbb{Z}^d : \|x - y\|_1 = 1 \} \qquad \text{"edge set"}$$

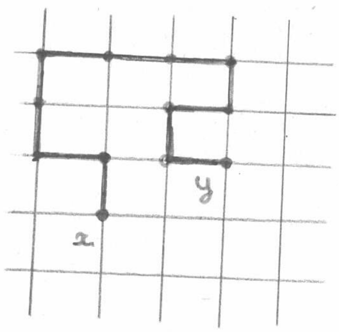
Notation:
- For $x, y \in \mathbb{Z}^d$, write $xy = \{x, y\}$.
- If $xy \in E$ we say that x and y are neighbours and we write $x \sim y$.



$\rightarrow (\mathbb{Z}^d, E)$ is an infinite graph of degree $2d$.

Def: A path of length ℓ from a vertex x to a vertex y is a sequence $\gamma = (\gamma_0, \dots, \gamma_\ell)$ of distinct vertices s.t.

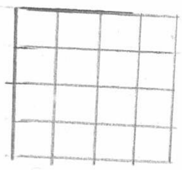
$$\gamma_0 = x, \quad \gamma_1 = y \quad \text{and} \quad \forall i \in \{1, \dots, \ell\} \ \gamma_i \gamma_{i-1} \in E.$$



Remark: $\|x-y\|_1 = \min \{ \text{length}(\gamma),\ \gamma \text{ path from } x \text{ to } y \}$

"graph distance between x and y."

Not. $\Lambda_n = \{-n, \dots, n\}^d$ "box of size n around 0"



Λ_2 on $\mathbb{Z}^2$

Def: Let $S \subset \mathbb{Z}^d$. We define

- $\partial S := \{ x \in S : \exists y \in \mathbb{Z}^d \setminus S\ \ y \sim x \}$ "vertex boundary of S"
- $\Delta S := \{ xy \in E : x \in S,\ y \notin S \}$ "edge boundary of S".

1.2. PERCOLATION CONFIGURATIONS

(bond) percolation configuration: $\omega = (\omega(e))_{e \in E} \in \{0,1\}^E$.

Rk: $\{0,1\}^E \overset{\text{bij}}{\longleftrightarrow} \mathcal{P}(E)$

$\omega \longleftrightarrow E_\omega = \{ e : \omega(e) = 1 \}$

We often identify ω with the subgraph $G_\omega = (\mathbb{Z}^d, E_\omega)$.

Def: Let $\omega \in \{0,1\}^E$

- An edge $e \in E$ is said to be - open if $\omega(e) = 1$,
  - closed if $\omega(e) = 0$.
- A path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_\ell)$ is said to be open if $\forall i\ \omega(\gamma_{i-1}\gamma_i) = 1$.
- A cluster is a connected component of G_ω.

Notation: $C_x(\omega)$ = cluster containing x

13 PROBABILITY FORMALISM.

Goal: mathematical "object" describing a random element of $\{0,1\}^E$, where the components are obtained as independent p-biased coin flips.

Idea 1: random variable formalism.

$(\Omega, \mathcal{G}, \mathbb{P})$ abstract auxiliary probability space

$X : \Omega \longrightarrow \{0,1\}^E$ measurable s.t.

under $\mathbb{P}$, $(X(e))_{e \in E}$ are iid Bernoulli (p) r.v.

↳ $\boxed{\text{mathematical object : random variable } X.}$

advantage: flexibility, eg. we can add randomness

disadvantage: the parameter p is implicit

→ difficult to study the effect of p in this formalism.

Idea 2: explicit probability space

We directly equip the set of configurations with a probability structure → $(\{0,1\}^E, \mathcal{F}, P_p)$

- $\mathcal{F}$ is the product σ-algebra.
- $P_p = \prod_{e \in E} \mu_e$ product measure where $\underbrace{\mu_e = p\,\delta_1 + (1-p)\,\delta_0}_{\text{proba. measure on } \{0,1\}}$

↳ $\boxed{\text{mathematical object : } \omega \text{ element of } \{0,1\}^E \text{ (equiped with } (\mathcal{F}, P_p))}$

Relations between the two formalisms.

- If $(X(e))_{e\in E}$ are iid Bernoulli (p), then the law of $X = (X(e))$ is P_p.

- If one considers the second formalism, and the projections $X_e : \begin{array}{ccc} \{0,1\}^E & \longrightarrow & \{0,1\} \\ \omega & \longmapsto & \omega(e) \end{array}$

  Under P_p $(X_e)_{e\in E}$ are iid Bernoulli (p)

In this class we will mainly work with the second formalism where the effect of p is more explicit : we have

↳ one measured space $(\{0,1\}, \mathcal{F})$

↳ a family of probability measures. $(P_p)_{0\le p\le 1}$.

- Some example of events (ie. elements of $\mathcal{F}$.).

→ $\{\omega(e_1) = u_1, \dots \ \omega(e_k) = u_k\}$ $\quad e_1, \dots, e_k \in E \quad u_1, \dots, u_k \in \{0,1\}$

→ $\{x \leftrightarrow y\} = \{\exists \text{ open path from } x \text{ to } y\}$ $\quad x, y \in \mathbb{Z}^d$

→ $\{A \leftrightarrow B\} = \{\exists x\in A \ \exists y \in B \quad x \leftrightarrow y\}$ $\quad A, B \subset \mathbb{Z}^d$

→ $\{x \leftrightarrow \infty\} = \{x \text{ belongs to an infinite cluster}\}$ $\quad x \in \mathbb{Z}^d$

- Some examples of real random variables. (i.e. meas. fct: $\{0,1\}^E \to \mathbb{R} \cup \{+\infty\}$)

→ $w(e): \begin{cases} \{0,1\}^E \to \{0,1\} \\ \omega \mapsto \omega(e) \end{cases}$ "value of one edge".

→ $N: \begin{cases} \{0,1\}^E \to \mathbb{N} \cup \{+\infty\} \\ \omega \mapsto \text{number of infinite clusters in } \omega. \end{cases}$

- Reminder: P_p is characterized by

$$\forall e_1, \dots e_k \ \forall u_1, \dots u_k \quad P_p[\omega(e_1) = u_1, \dots, \omega(e_k) = u_k] = p^{|u|} \times (1-p)^{k-|u|}$$

where $|u| = \sum_{i=1}^{k} u_i$.

- Notation: E_p → expectation associated to P_p

2 PROPERTIES:

21. MONOTONICITY

Questions: We expect $P_p[x \leftrightarrow y]$ increasing in p.

→ What is the property of $A = \{x \leftrightarrow y\}$ implying this fact.

→ How to prove it?

Equip $\{0,1\}^E$ with the product ordering

$$\omega \leq \eta \iff \forall e \in E \quad \omega(e) \leq \eta(e).$$

Def: An event $A \in \mathcal{F}$ is <u>increasing</u> if

$$\left.\begin{array}{l} \omega \leq \eta \\ \omega \in A \end{array}\right\} \Rightarrow \eta \in A.$$

A is decreasing if A^c decreasing.

Ex: $\{x \leftrightarrow y\}$, $\{|C_x| \geq 10\}$ are increasing.

$\{|C_x| = 10\}$ is neither increasing or decreasing.

Rk: if A, B are increasing then $A \cap B$, $A \cup B$ are increasing.

Def: A function $f : \{0,1\}^E \longrightarrow \mathbb{R}$ is <u>increasing</u> if

$$\omega \leq \eta \Rightarrow f(\omega) \leq f(\eta).$$

It is decreasing if

$$\omega \leq \eta \Rightarrow f(\omega) \geq f(\eta).$$

Ex: $f(\omega) = |C_x(\omega)|$ is increasing.

Rk: A increasing $\Leftrightarrow$ $\mathbb{1}_A$ increasing.

Prop (i) Let $A \in \mathcal{F}$ be an increasing event, then

$$p \longmapsto \mathbb{P}_p[A] \text{ is non-decreasing.}$$

(ii) Let $f : \{0,1\}^E \longrightarrow \mathbb{R}$ measurable increasing, bounded or ≥ 0. Then

$$p \longmapsto \mathbb{E}_p[f] \text{ is non decreasing.}$$

Proof: It suffices to prove (ii). (i) follows by applying (ii) with $f = \mathbb{1}_A$.

We use a monotone coupling. Let $(U_e)_{e \in E}$ be iid uniform in $[0,1]$.

Define for every $p \in [0,1]$ $X_p(e) = \mathbb{1}_{U_e \leq p}$.

We have $p \leq p' \Rightarrow X_p \leq X_{p'}$ a.s.

$$\Rightarrow f(X_p) \leq f(X_{p'}) \text{ a.s.}$$

$$\Rightarrow \underbrace{\mathbb{E}[f(X_p)]}_{E_p[f]} \leq \underbrace{\mathbb{E}[f(X_{p'})]}_{E_{p'}[f]}$$

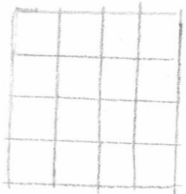
■

Appli: $P_p[x \leftrightarrow y]$, $P_p[x \leftrightarrow \infty]$, $E_p[|C_0|]$ are non decreasing in p.

2.2 RUSSO'S FORMULA

We consider percolation on a finite graph $G = (V, E)$.
(same def. as on $(\mathbb{Z}^d, E)$)

Ex: $G =$ subgraph induced by Λ_n.



Def. Let $A \subset \{0,1\}^E$. We say that $e \in E$ is pivotal for A in ω if

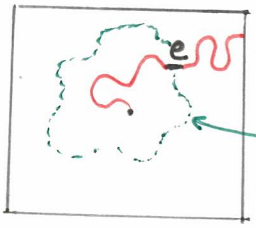
$$\mathbb{1}_A(\omega) \neq \mathbb{1}_A(\omega^e)$$

where $\omega^e(f) := \begin{cases} \omega(f) & f \neq e \\ 1-\omega(f) & f = e \end{cases}$

Rk: • The event $\{e \text{ is piv. for } A\}$ $\left(= \{\omega : \mathbb{1}_A(\omega) \neq \mathbb{1}_A(\omega^e)\} \right)$ is

- measurable with respect to $(\omega(f))_{f \neq e}$,
- independent of $\omega(e)$.

Ex: on Λ_m, $A = 0 \longleftrightarrow \partial\Lambda_m$.



"blocking closed surface"

Diagrammatic representation of the event $\{e$ is pivotal for $A\}$.

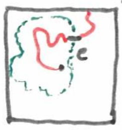
Prop: Let $A \in \{0,1\}^E$ increasing (recall that $|E| < \infty$). Then

$$\boxed{\frac{d}{dp} \mathbb{P}_p[A] = \sum_{e \in E} \mathbb{P}_p[e \text{ is piv. for } A]}$$

Rk: $\mathbb{P}_p[A] = \underset{\substack{\uparrow \\ \text{"finite sum"}}}{\sum_{\omega \in A}} p^{|\omega|} (1-p)^{|E| - |\omega|}$ polynomial in p, in particular C^∞.

(where $|\omega| = \sum_{e \in E} \omega(e)$)

Rk: Russo's formula gives a "geometric" interpretation of the derivative of connection probabilities. For example

$$\frac{d}{dp} \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] = \sum_{e \subset \Lambda_n} \mathbb{P}_p \Bigg[$$  $$\Bigg].$$

Proof: Let $E = \{e_1, \dots, e_k\}$.

Define $\forall p_1, \dots, p_k \in [0,1]$ $\forall \omega \in \{0,1\}^E$

$$\mathbb{P}_{p_1, \dots, p_k}[\{\omega\}] = \prod_{i=1}^{k} p_i^{\omega_i} (1-p_i)^{1-\omega_i}. \quad \text{where } \omega_i := \omega(e_i).$$

(Rk: $\mathbb{P}_p = \mathbb{P}_{p, \dots, p}$)

Define $f(p_1, \dots, p_k) = P_{p_1, \dots, p_k}[A]$

$$\frac{d}{dp} P_p[A] = \frac{d}{dp} f(p, \dots, p) = \sum_{i=1}^{k} \frac{\partial}{\partial p_i} f(p, \dots, p).$$

Fix $i \in \{1, \dots, k\}$

$$A = \underbrace{\left(\{e_i \text{ not piv. for } A\} \cap A \right)}_{=: A'} \overset{\text{"disjoint"}}{\cup} \underbrace{\left(\{e_i \text{ piv for } A\} \cap A \right)}_{\underset{\text{"A increasing"}}{=} \{e_i \text{ piv for } A\} \cap \{w(e_i)=1\}}$$

$$= A' \cup \left(\{e_i \text{ piv for } A\} \cap \{w(e_i)=1\} \right)$$

meas. w.r.t. $(\omega(e_j))_{j \neq i}$

$$P_{p_1, \dots, p_k}[A] = P_{p_1, \dots, p_k}[A'] + P_{p_1, \dots, p_k}[e_i \text{ piv for } A] \cdot p_i$$

do not depend on p_i

"affine function of p_i"

Hence $\frac{\partial}{\partial p_i} f(p_1, \dots, p_k) = P_{p_1, \dots, p_k}[e_i \text{ piv for } A]$ ∎

2.3 HARRIS FKG INEQUALITY (FKG: Fortuin, Kasteleyn, Ginibre)

$G = (\mathbb{Z}^d, E)$

<u>Intuition</u>: Knowing that a certain connection event occurs should "help" other connections to occur.

We expect
- $\forall x, y, z, t \in \mathbb{Z}^d \quad P_p[z \leftrightarrow t \mid x \leftrightarrow y] \geq P_p[z \leftrightarrow t]$
- and more generally $\forall A, B \uparrow \quad P_p[A \mid B] \geq P_p[A]$

Prop. i) Let $A, B \uparrow$ events, then $\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\,\mathbb{P}_p[B]$

ii) Let $X, Y : \Omega \to \mathbb{R}$ be two bounded increasing random variables. Then

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\,\mathbb{E}_p[Y].$$

Rk: Holds in more general contexts $\to$ dependent models, general product space ...

Proof. It suffices to prove (ii). (i) follows by considering $X = \mathbb{1}_A$ and $Y = \mathbb{1}_B$. Order $E = \{e_1, e_2, \cdots\}$ and write $\omega_i = \omega(e_i)$

Finite volume.

We first prove by induction on $n \geq 1$ that

$(\mathcal{P}_n)$ $\quad \forall f, g : \{0,1\}^n \longrightarrow \mathbb{R}$ increasing

$$\mathbb{E}_p[f(\omega_1, \dots, \omega_n)\, g(\omega_1, \dots, \omega_n)] \geq \mathbb{E}_p[f(\omega_1, \dots, \omega_n)]\, \mathbb{E}_p[g(\omega_1, \dots, \omega_n)].$$

$n=1$ Let $f, g : \{0,1\} \to \mathbb{R}$ increasing. WLOG, we can assume that $f(0) = g(0) = 0$ since adding a constant to f and/or g does not change the inequality. In such case, we have $f(1) \geq 0$ and $g(1) \geq 0$ since f, g are increasing.

Hence $\mathbb{E}_p[f(\omega_1)\, g(\omega_1)] - \mathbb{E}_p[f(\omega_1)]\, \mathbb{E}_p[g(\omega_1)]$

$$= p\, f(1)\, g(1) - p^2 f(1)\, g(1)$$

$$\geq 0$$

Now let $n \geq 1$ and assume that $(\mathcal{P}_n)$ holds.
Let $f, g : \{0,1\}^{n+1} \longrightarrow \mathbb{R}$ increasing.

$$E_p[f(\omega_1,\dots,\omega_{m+1})\, g(\omega_1,\dots,\omega_{m+1})]$$

$$= p\, E_p[f(\omega_1,\dots,\omega_m,1)\, g(\omega_1,\dots,\omega_m,1)] + (1-p)\, E_p[f(\omega_1,\dots,\omega_m,0)\, g(\omega_1,\dots,\omega_m,0)]$$

$$\overset{\mathcal{P}_m}{\geq} p\, \underbrace{E_p[f(\omega_1,\dots,\omega_m,1)]}_{=:f_1(1)}\, \underbrace{E_p[g(\omega_1,\dots,\omega_m,1)]}_{=:g_1(1)} + (1-p)\, \underbrace{E_p[f(\omega_1,\dots,\omega_m,0)]}_{=:f_1(0)}\, \underbrace{E_p[g(\omega_1,\dots,\omega_m,0)]}_{=:g_1(0)}.$$

$$= E_p[f_1(\omega_1)\, g_1(\omega_1)] \overset{\mathcal{P}_1}{\geq} E_p[f_1(\omega_1)]\, E_p[g_1(\omega_1)].$$

This proves $(\mathcal{P}_{m+1})$ since

$$\begin{aligned} E_p[f_1(\omega_1)] &= p\, f_1(1) + (1-p)\, f_1(1) \\ &= p\, E_p[f(\omega_1,\dots,\omega_m,1)] + (1-p)\, E_p[f(\omega_1,\dots,\omega_m,0)] \\ &= E_p[f(\omega_1,\dots,\omega_{m+1})]. \end{aligned}$$

and equivalently $E_p[g_1(\omega_1)] = E_p[g(\omega_1,\dots,\omega_{m+1})]$

<u>Infinite volume.</u>

Let $X, Y : \{0,1\}^E \to \mathbb{R}$ be two increasing bounded random variables.

Let $X_m = E_p[X \mid \omega_1,\dots,\omega_m]$, $Y_m = E_p[Y \mid \omega_1,\dots,\omega_m]$.

For every $m \geq 1$, we have, by $\mathcal{P}_m$,

$$E_p[X_m Y_m] \geq E_p[X_m]\, E_p[Y_m].$$

By the martingale convergence theorem*, we have

$$X_m \to X \text{ and } Y_m \to Y \text{ in } L^2 \text{ and } L^1,$$

and we obtain

$$E_p[XY] \geq E_p[X]\, E_p[Y]$$

by taking the limit in the equation above as m tends to infinity.

* see e.g. GRIMMETT STIRZAKER p. 484 (3rd edition)
or WALTERS (Probability with martingales) p. 134.

Corollary:

If A, B are decreasing, then
$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\,\mathbb{P}_p[B].$$
If A is increasing and B is decreasing, then
$$\mathbb{P}_p[A \cap B] \leq \mathbb{P}_p[A]\,\mathbb{P}_p[B].$$

Corollary: (square-root trick)

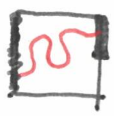
Let $A_1, \dots, A_k$ be k increasing events, $k \geq 1$. Let $\varepsilon > 0$.

If $\mathbb{P}_p[A_1 \cup \dots \cup A_k] \geq 1 - \varepsilon$,

Then $\max\limits_{1 \leq i \leq k} \mathbb{P}_p[A_i] \geq 1 - \varepsilon^{1/k}$.

Proof:
$$\mathbb{P}_p[A_1 \cup \dots \cup A_k] = 1 - \mathbb{P}_p[A_1^c \cap \dots \cap A_k^c]$$
$$\underset{+\text{induction}}{\overset{\text{FKG}}{\leq}} 1 - \mathbb{P}_p[A_1^c] \cdots \mathbb{P}_p[A_k^c]$$
$$\leq 1 - \left(1 - \max_{1 \leq i \leq k} \mathbb{P}_p[A_i^c]\right)^k$$
ie $\max\limits_{1 \leq i \leq k} \mathbb{P}_p[A_i] \geq 1 - \left(1 - \mathbb{P}_p[A_1 \cup \dots \cup A_k]\right)^{1/k}$ ∎

Application: for percolation on $\mathbb{Z}^2$

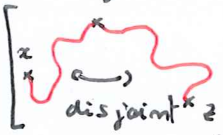
$\mathbb{P}_p\Big[$  $\Big] \geq 1 - \varepsilon \implies \mathbb{P}_p\Big[$  $\Big] \geq 1 - \sqrt{\varepsilon}$.

open path from left to right.

open path from left to right that ends on the upper half segment of the right side.

2.4. BK - REIMER INEQUALITY.

$G=(V,E)$ finite graph.

Motivation: We expect $\mathbb{P}_p\Big[$  $\Big] \leq \mathbb{P}_p[x \leftrightarrow y]\,\mathbb{P}_p[y \leftrightarrow z]$.

Def: Let A be an event. Let $\omega \in \{0,1\}^E$.

A set $I \subset E$ is a witness of A in ω (I wit. A in ω) if

$\omega \in A$ and $\forall \omega' \in \{0,1\}^E$, $(\omega_{|I} = \omega'_{|I}) \Rightarrow (\omega' \in A)$.

Ex. An open path from x to y in ω is a witness of $x \leftrightarrow y$ in ω.

. If $\omega \in A$, then we always have E wit. A in ω.

Question: What would be a witness for $|C_0| = 5$? $|C_0| \geq 5$? $|C_0| \leq 5$?

Exercise: Let $A \uparrow$, $\omega \in A$. Prove that there exists a wit. I for A in ω s.t. $\forall e \in I \;\; \omega(e) = 1$.

Def: Let A, B be two events. Define

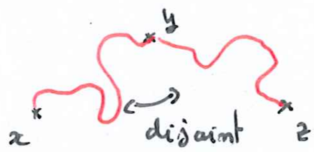
"may depend on ω"

$$A \circ B = \left\{ \omega \in \{0,1\}^E : \exists\, I, J \text{ disjoint s.t. } \begin{matrix} I \text{ wit. } A \text{ in } \omega \\ J \text{ wit. } B \text{ in } \omega \end{matrix} \right\}.$$

When $A \circ B$ occurs, we say that A and B occur disjointly.

Examples

. $\{e \text{ is open}\} \circ \{f \text{ is open}\} = \begin{cases} \emptyset & \text{if } e = f \\ \{e, f \text{ open}\} & \text{if } e \neq f. \end{cases}$

. $\{x \leftrightarrow y\} \circ \{y \leftrightarrow z\}$ " = " 

<u>Rk</u>: We always have $A \circ B \subset A \cap B$.

Sometimes, we may have $A \circ B = A \cap B$. For example, this equality holds if A and B depend on disjoint set of edges, or if A is $\uparrow$ and B is $\downarrow$. (exercice).

<u>Exercise</u>: Let $A, B \uparrow$. prove that

$$A \circ B = \left\{ \omega : \exists\, I, J \text{ disjoint } \underline{\text{open}} \text{ s.t. } \begin{matrix} I \text{ wit. } A \text{ in } \omega \\ J \text{ wit. } B \text{ in } \omega \end{matrix} \right\}.$$

"a set $I \subset E$ is open if all its edges are open"

<u>Thm: (BK - Reimer inequality)</u>

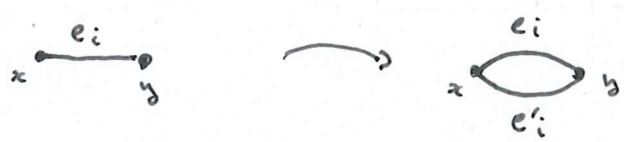
Let A, B be two events (depending on finitely many edges)
Then

$$\boxed{\mathbb{P}_p[A \circ B] \leq \mathbb{P}_p[A]\,\mathbb{P}_p[B].}$$

<u>Rk</u>: Proved by Van den Berg and Kesten for increasing events, extended to general events by Reimer using a different approach. In this course, we present the proof for increasing events.

<u>Proof</u>: Let $A, B \subset \{0,1\}^E$ increasing*. Write $E = \{e_1, \dots, e_m\}$.

We use a construction where the edges are "duplicated": for each edg e_i we add a parallel edge e_i'



* in all the proof the sets witnessing A or B are always assumed to be open.

We then consider independent percolation on the resulting graph. This amounts to consider two copies of the space,

$$\omega = (\omega_1, \dots, \omega_m) \quad \text{and} \quad \omega' = (\omega'_1, \dots, \omega'_m)$$

where $\omega_i = \omega(e_i)$, $\omega'_i = \omega'(e_i)$. We write $\bar{\omega} = (\omega, \omega')$ and $\bar{\mathbb{P}}_p$ the corresponding product measure. Introduce for $0 \le i \le m$

$$\omega^{(i)} = (\omega'_1, \dots, \omega'_i, \omega_{i+1}, \dots, \omega_m).$$

interpolating between $\omega^{(0)} = \omega$ and $\omega^{(m)} = \omega'$.

Let

$$\bar{A}_i = \{\bar{\omega} : \omega^{(i)} \in A\} \quad \text{and} \quad \bar{B} = \{\bar{\omega} : \omega \in B\}.$$

We write $\bar{E} = E \cup E'$ for the set of all duplicated edges. The disjoint occurrence is well define on $\{0,1\}^{\bar{E}}$ and we have

$$\bar{\mathbb{P}}_p[\bar{A}_m \circ \bar{B}] = \bar{\mathbb{P}}_p[\bar{A}_m]\, \bar{\mathbb{P}}_p[B] = \mathbb{P}_p[A]\, \mathbb{P}_p[B]$$

(since $\bar{A}_0$ and $\bar{B}$ depend on disjoint set of edges)

and

$$\bar{\mathbb{P}}_p[\bar{A}_0 \circ \bar{B}] = \mathbb{P}_p[A \circ B]$$

(since $\bar{A}_0$ and $\bar{B}$ are only defined in term of ω).

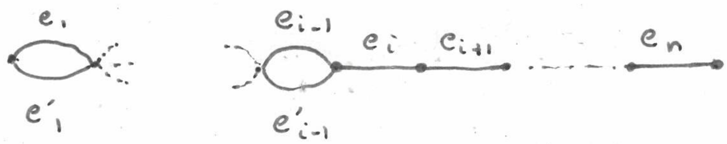
Hence, BK inequality can be rewritten as.

$$\bar{\mathbb{P}}_p[\bar{A}_0 \circ \bar{B}] \le \mathbb{P}_p[\bar{A}_m \circ \bar{B}].$$

It suffices to show that for every $1 \le i \le m$

$$\bar{\mathbb{P}}_p[\bar{A}_{i-1} \circ \bar{B}] \le \mathbb{P}[\bar{A}_i \circ \bar{B}].$$

$\overline{A_{i-1}} \circ \overline{B}$



$\overline{A_i} \circ \overline{B}$

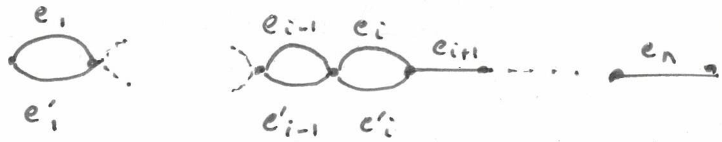


Illustration of the edges "used" to define
$\overline{A_{i-1}} \circ \overline{B}$ and $\overline{A_i} \circ \overline{B}$

For $\bar{\omega} \in \{0,1\}^{\bar{E}}$, and $\underset{(\delta_0,\delta_1)}{\underset{\shortparallel}{\delta}} \in \{0,1\}^2$, define

$$\forall f \in \bar{E} \qquad \bar{\omega}^{\delta}(f) = \begin{cases} \bar{\omega}(f) & f \notin \{e_i, e_i'\} \\ \delta_0 & f = e_i \\ \delta_1 & f = e_i' \end{cases}$$

Let $\mathcal{D}(\bar{\omega}) = \{ \delta \in \{0,1\}^2 : \bar{\omega}^{\delta} \in \overline{A_i} \circ \overline{B} \}$

(notice that $\mathcal{D}(\bar{\omega})$ measurable with resp. to $(\omega_j)_{j \neq i}, (\omega_j')_{j \neq i}$)

Since $\overline{A_i} \circ \overline{B}$ is increasing, the set $\mathcal{D}$ must be increasing

$\rightarrow \quad \mathcal{D}(\bar{\omega}) \in \underbrace{\{ \{0,1\}^2, \{0,1\}^2 \setminus \{(0,0)\}, \{1\} \times \{0,1\}, \{0,1\} \times \{1\}, \{(1,1)\}, \emptyset \}}_{\text{"admissible"}}$

By using the decomposition

$$\bar{P}_p[C] = \sum_D \bar{P}_p[C \mid \mathcal{D} = D] \; \bar{P}_p[\mathcal{D} = D]$$

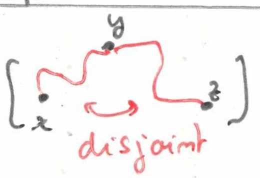
for $C = \overline{A_{i-1}} \circ \overline{B}$ and $C = \overline{A_i} \circ \overline{B}$, it suffices to prove $P_p[\overline{A_{i-1}} \circ \overline{B} \mid \mathcal{D} = D] \leq P_p[\overline{A_i} \circ \overline{B} \mid \mathcal{D} = D]$

for every admissible D, which we now do in a case by case manner:

D	$P_p[\overline{A_{i-1}} \circ \overline{B} \mid \mathcal{D} = D]$	$P_p[\overline{A_i} \circ \overline{B} \mid \mathcal{D} = D]$
$\{0,1\}^2$	1	1
$\{0,1\}^2 \setminus \{(0,0)\}$	p	$p + p(1-p)$
$\{0,1\} \times \{1\}$	p	p
$\{1\} \times \{0,1\}$	p	p
$\{1,1\}$	0	p^2
$\emptyset$	0	0

This completes the proof. $\blacksquare$

Applications for percolation on $(\mathbb{Z}^d, E)$

Appli 1: $P_p\Big[$ 

$\Big] \leq P_p[x \longleftrightarrow y]\, P_p[y \longleftrightarrow z]$

Not: for $S \subset \mathbb{Z}^d$, write $A \overset{S}{\longleftrightarrow} B$ for the events that there exists a path from A to B, all the vertices of which belong to S.

"proof": Let $m \geq 1$. By BK-inequality,

$$P_p\left[\{x \overset{\Lambda_m}{\longleftrightarrow} y\} \circ \{y \overset{\Lambda_m}{\longleftrightarrow} z\}\right] \leq P_p\left[x \overset{\Lambda_m}{\longleftrightarrow} y\right] P_p\left[y \overset{\Lambda_m}{\longleftrightarrow} z\right]$$

and we obtain the result by letting m tend to infinity. ■

2.5 INVARIANCE, MIXING PROPERTY AND ERGODICITY.

$G = (\mathbb{Z}^d, E)$.

$\mathbb{Z}^d$ (additive group) acts on . $\mathbb{Z}^d$ by translation $z \cdot x = z + x$

. E : $z \circ \{x, y\} = \{z+x, z+y\}$.

. $\{0,1\}^E$: $(z \cdot \omega)(\{x,y\}) = \omega(\{x-z, y-z\})$

. $\mathcal{F}$: $z \cdot A = \{z \cdot \omega, \omega \in A\}$.

NB: (e open in ω) $\Longleftrightarrow$ ($z \cdot e$ open in $z \cdot \omega$)

Ex: If $A = x \longleftrightarrow y$, then $z \cdot A = \{z + x \longleftrightarrow z + y\}$.

Prop. For every event A and every $z \in \mathbb{Z}^d$, we have

$$P_p[z \cdot A] = P_p[A]. \quad \text{"}P_p \text{ is invariant"}$$

Proof: True for cylinder events. Conclude with monotone class theorem.

Appli: $P_p[0 \longleftrightarrow \infty] = P[z \longleftrightarrow \infty]$

Prop. [MIXING PROPERTY]

Let A, B be two events. Then

$$\lim_{|z|\to\infty} \mathbb{P}_p[A \cap z \cdot B] = \mathbb{P}_p[A]\,\mathbb{P}_p[B]$$

Proof: Let $\varepsilon > 0$. Choose $A_\varepsilon, B_\varepsilon$ depending on finitely many edges such that

$$\mathbb{P}_p[A \,\Delta\, A_\varepsilon] \leq \varepsilon \quad \text{and} \quad \mathbb{P}_p[B \,\Delta\, B_\varepsilon] \leq \varepsilon.$$

"symmetric difference"

By independence, if $|z|$ is large enough, we have

$$\mathbb{P}_p[A_\varepsilon \cap z \cdot B_\varepsilon] = \mathbb{P}_p[A_\varepsilon] \cdot \mathbb{P}_p[z \cdot B_\varepsilon]$$

$$= \mathbb{P}_p[A_\varepsilon]\,\mathbb{P}_p[B_\varepsilon]$$

invariance.

Therefore, if $|z|$ large enough

$$\mathbb{P}_p[A \cap z \cdot B] \leq \mathbb{P}_p[A_\varepsilon \cap z \cdot B_\varepsilon] + 2\varepsilon$$

$$= \mathbb{P}_p[A_\varepsilon]\,\mathbb{P}_p[B_\varepsilon] + 2\varepsilon$$

$$\leq \mathbb{P}_p[A]\,\mathbb{P}_p[B] + 4\varepsilon.$$

Equivalently, $\mathbb{P}_p[A \cap z \cdot B] \geq \mathbb{P}_p[A]\,\mathbb{P}_p[B] - 4\varepsilon$, which concludes the proof. ∎

Application : $P[0 \leftrightarrow \infty, z \leftrightarrow \infty] \xrightarrow[|z| \to \infty]{} P_p[0 \leftrightarrow \infty]^2 \quad (= \theta(p)^2)$

Prop. [ERGODICITY]

Let A be an invariant event (ie $\forall z \in \mathbb{Z}^d \quad z.A = A$).
Then
$$P_p[A] \in \{0,1\}.$$

Proof: By invariance of A, $P_p[A] = P[A \cap z \cdot A]$. Hence

$$P_p[A] = \lim_{|z| \to \infty} P_p[A \cap z.A] \underset{\substack{\uparrow \\ \text{Mixing}}}{=} P_p[A]^2 \qquad \blacksquare$$

Application: Let $N(\omega)$ be the number of disjoint infinite clusters in ω. Then $\forall k \in \mathbb{N} \cup \{\infty\}$

$$P_p[N = k] \in \{0,1\}.$$

CHAPTER 2 :

SUBCRITICAL PERCOLATION.

$G = (\mathbb{Z}^d, E)$, $d \geq 2$.

1. PHASE TRANSITION

Not. : $\theta(p) = \mathbb{P}_p[0 \longleftrightarrow \infty]$, $\theta_n(p) = \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n]$.

Rk: $\theta : [0,1] \to [0,1]$ is non-decreasing.

Exercise: Prove that θ is right continuous. (Hint: use $\theta = \lim_{n\to\infty} \theta_n$)

Def. The critical parameter for Bernoulli percolation is defined by

$$p_c = \sup\{p \in [0,1] : \theta(p) = 0\}.$$

Rk: We have already seen in the introduction that $0 < p_c < 1$.

Question: We know that for $p < p_c$ $\theta_n(p) \xrightarrow[n\to\infty]{} 0$. At which speed?

2. EXPONENTIAL DECAY

In this section, our goal is to prove the following theorem.

Thm [AIZENMAN-BARSKY, MENSHIKOV, '87]

(i) $\forall p < p_c$ $\exists c = c(p)$ s.t. $\forall n \geq 1$

$$\theta_n(p) \leq e^{-cn}.$$

(ii) $\forall p \geq p_c$ $\theta(p) \geq \frac{1}{2}(p - p_c)$. "mean field lower bound".

The bound (ii) is sharp in the following sense. For Bernoulli percolation on a tree or on $\mathbb{Z}^d$, $d \geq 6$, we expect $\theta(p) \underset{p \to p_c}{\sim} C(p-p_c)$.
This is known for the tree, and the upper bound $\theta(p) \leq C(p-p_c)$ is known for $\mathbb{Z}^d$, $d \geq 11$. On $\mathbb{Z}^2$ we will see that $\theta(p) \geq c(p-p_c)^{\gamma}$ $\gamma < 1$.

Def: Let $S \subset \mathbb{Z}^d$ finite s.t. $0 \in S$. Introduce

$$\phi_p(S) = \sum_{xy \in \Delta S} p \cdot \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x]$$

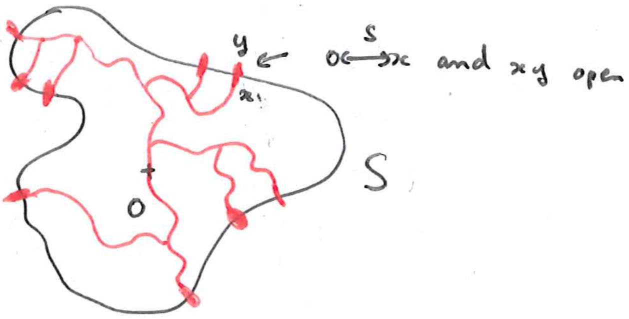
Convention: if $0 \notin S$ we set $\phi_p(S) = 0$.

Geometric interpretation:

$$\phi_p(S) = \sum_{xy \in \Delta S} \mathbb{P}_p[xy \text{ open}]\, \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x]$$

$$\overset{\text{indep.}}{=} \sum_{xy \in \Delta S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x,\ xy \text{ open}]$$

$$= \mathbb{E}\Big[\sum_{xy \in \Delta S} \mathbb{1}_{0 \overset{S}{\longleftrightarrow} x,\ xy \text{ open}}\Big]$$



$$\phi_p(S) = \mathbb{E}_p\Big[\text{"number of open edges through which one can exit } S \text{ starting at } 0\text{"}\Big]$$

Lemma 1. Let $S \subset \mathbb{Z}^d$ finite s.t. $0 \in S$. Assume that

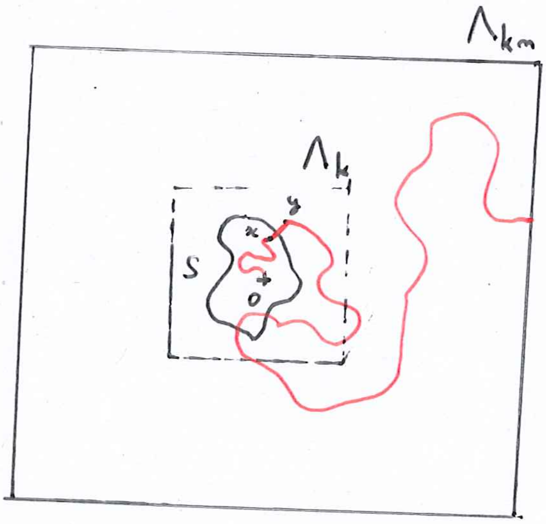
$$\varphi_p(S) < 1$$

Then there exists $c > 0$ s.t.

$$\forall n \geq 1 \quad \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}.$$

Proof: Let k large enough s.t. $S \subset \Lambda_k$.

If $0 \longleftrightarrow \partial\Lambda_{kn}$ occurs, then there exists an edge xy at the boundary of S s.t $\{0 \longleftrightarrow x, xy \text{ open}\}$ and $\{y \longleftrightarrow \partial\Lambda_{kn}\}$ occur disjointly. (To see this, consider the first traversed edge xy at the boundary of S, when following an open path from 0 to $\partial\Lambda_{kn}$).



$\{0 \longleftrightarrow x, xy \text{ open}\}$ and $\{y \longleftrightarrow \partial\Lambda_{kn}\}$ occur disjointly when there exists an open path from 0 to $\partial\Lambda_{kn}$.

By the union bound and BK-inequality, we find

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{km}] \leq \sum_{xy \in \Delta S} \mathbb{P}_p[\{0 \overset{S}{\longleftrightarrow} x, xy \text{ open}\} \circ \{y \longleftrightarrow \partial\Lambda_{km}\}]$$

$$\overset{BK}{\leq} \sum_{xy \in \Delta S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x, xy \text{ open}] \underbrace{\mathbb{P}_p[y \longleftrightarrow \partial\Lambda_{km}]}_{\leq \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{k(m-1)}]}$$

"translation invariance" (see 2.5)

$$\leq \phi_p(S) \cdot \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{k(m-1)}]$$

By induction, we obtain, for every $n \geq 1$

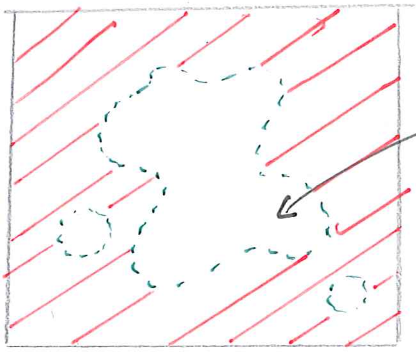
$$P[0 \longleftrightarrow \partial\Lambda_{km}] \leq \phi_p(S)^n$$

■

<u>Rk</u>: The percolation cluster C_0 is "smaller" than a subcritical branching process when $\phi_p(S) < 1$. $\phi_p(S)$ can be interpreted as the "progeny" of C_0 at the boundary of S.

<u>Lemma 2</u>. Consider the random set $\mathcal{S}_m = \{x \in \Lambda_m : x \not\longleftrightarrow \partial\Lambda_m\}$. Then for every fixed $m \geq 1$ and every $p \in [0,1]$,

$$\theta_m'(p) \geq \frac{1}{p(1-p)} \mathbb{E}_p[\phi_p(\mathcal{S}_m)].$$



$\mathcal{S}_n$ is the the set of points that are not connected to $\partial\Lambda_n$

it can be seen as the complement of all the clusters touching $\partial\Lambda_n$

<u>Proof:</u> Let E_n be the edges between vertices in Λ_n. Apply Russo's to $A = 0 \leftrightarrow \partial\Lambda_n$ to get.

$$\theta_n'(p) = \sum_{e\in E_n} \mathbb{P}_p[e \text{ is piv. for } A]$$

$$\underset{\substack{\uparrow\\ \{e \text{ is piv}\} \text{ indep. of } \omega(e)}}{=} \frac{1}{1-p} \sum_{e\in E_n} \mathbb{P}_p[e \text{ piv. for } A,\ e \text{ closed}]$$

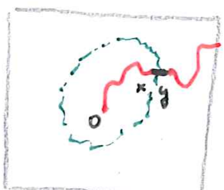
$$= \frac{1}{1-p} \sum_{e\in E_n} \mathbb{P}_p[\{e \text{ piv. for } A\} \cap A^c]$$

Now we use the partition $A^c = \bigsqcup_{\substack{S\subset\Lambda_n:\\ 0\in S}} \{\mathcal{S}_n = S\}$.

$$\theta_n' = \frac{1}{1-p} \sum_{\substack{S\subset\Lambda_n\\ 0\in S}} \sum_{e\in E_n} \mathbb{P}_p[e \text{ piv for } A,\ \mathcal{S}_n = S]$$

Observation: An edge e is pivotal for A if

- one of its extremity x is connected to 0
- the other extremity y is connected to $\partial\Lambda_n$
- 0 is not connected to Λ_n in ω_e.



On the event $\mathcal{S} = S$, one sees that an edge e is piv if and only if

- $e \in \Delta S$
- one extremity of e is connected to 0 in S.

Hence,

$$\theta_n'(p) = \frac{1}{1-p} \sum_{\substack{S \subset \Lambda_n \\ 0 \in S}} \sum_{xy \in \Delta S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x, \mathcal{S}_n = S]$$

The event $\mathcal{S}_n = S$ is measurable with respect to the edges adjacent to at least one edge in S^c, while the event $0 \overset{S}{\longleftrightarrow} x$ is measurable with respect to the edges with both extremities in S. Hence, these two events are independent, and we obtain.

$$\theta_n'(p) = \frac{1}{1-p} \sum_{\substack{S \subset \Lambda_n \\ 0 \in S}} \underbrace{\Big(\sum_{xy \in \Delta S} \mathbb{P}_p[0 \longleftrightarrow x] \Big)}_{= \frac{1}{p} \phi_p(S)} \mathbb{P}_p[\mathcal{S}_n = S]$$

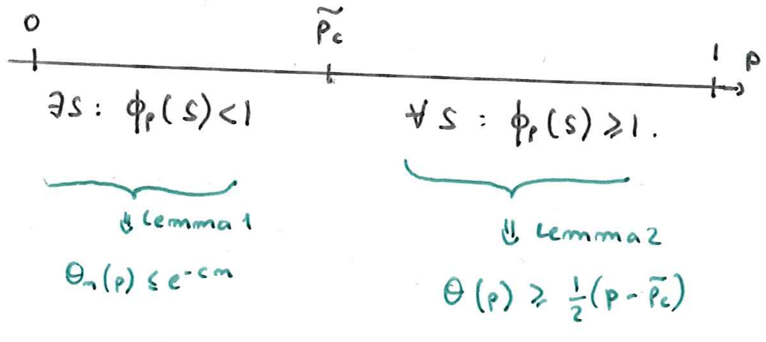
$$= \frac{1}{p(1-p)} \mathbb{E}_p[\phi_p(\mathcal{S}_n)] .$$

■

<u>Proof of the theorem.</u>

A set S is always assumed to satisfy $|S| < \infty$ $0 \in S$.

Introduce $\tilde{p}_c = \sup\{p \in [0,1] : \exists S \subset \mathbb{Z}^d \; \phi_p(S) < 1\}$.



By lemma 1, we have $\forall p < \tilde{p_c}$ $\exists c > 0 : \forall n$ $\theta_n(p) \leq e^{-cn}$.

By lemma 2, we have $\forall p \geq \tilde{p_c}$ and $\forall n$

$$\theta_n' \geq \frac{1}{p(1-p)} \mathbb{E}_p[\mathbb{1}_{0 \in S_n}]$$
$$\geq 1 - \theta_n$$

Fix $p > \tilde{p_c}$. If $\theta_n(p) \geq \frac{1}{2}$ we also have $\theta_n(p) \geq \frac{1}{2}(p - \tilde{p_c})$. Otherwise we have $\forall q \in [\tilde{p_c}, p]$ $\theta_n' \geq \frac{1}{2}$ and we obtain $\theta_n(p) \geq \frac{1}{2}(p - p_c)$ by integrating between $\tilde{p_c}$ and p.

Conclusion: $\forall n$ $\forall p > \tilde{p_c}$ $\theta_n(p) \geq \frac{1}{2}(p - \tilde{p_c})$ and we obtain $\forall p > \tilde{p_c}$ $\theta(p) \geq \frac{1}{2}(p - \tilde{p_c})$ by letting n tend to infinity. This concludes that $p_c = \tilde{p_c}$, and finishes the proof of the theorem.

Remarks:

1. Lemma 2 actually gives $\forall p > \tilde{p_c}$ $\theta_n' \geq \frac{1}{p(1-p)}(1 - \theta_n)$, which can be integrated between $\tilde{p_c}$ and p to prove the stronger bound $\theta(p) \geq \frac{1}{p(1-\tilde{p_c})}(p - \tilde{p_c})$.

2. $\{p : \exists S \ \phi_p(S) < 1\} = \bigcup_{\substack{S \subset \mathbb{Z}^d \text{ finite} \\ 0 \in S}} \{p : \phi_p(S) < 1\}$ is open.

   In particular p_c does not belong to this set and we have $\forall S \subset \mathbb{Z}^d$ finite s.t. $0 \in S$, $\phi_{p_c}(S) \geq 1$.

   This implies $\mathbb{E}_{p_c}[|C_0|] \geq \sum_n \phi_p(\Lambda_n) = +\infty$.

3. Since $\phi_p(\{0\}) = 2dp$, we see that $p_c(d) \geq \frac{1}{2d}$.

3. CORRELATION LENGTH.

We have seen $\forall p < p_c \quad p^n \leq \theta_n(p) \leq e^{-cn} \quad (c > 0 \text{ constant})$

$\rightarrow$ can we obtain a more precise estimate?

<u>Thm</u> [Definition of the correlation length]

Let $e_1 = (1, 0, \dots, 0)$. Let $p \in [0,1]$. The quantity

$$\boxed{\xi(p) = \left(\lim_{n \to \infty} -\frac{1}{n} \log(\mathbb{P}_p[0 \longleftrightarrow ne_1]) \right)^{-1}}$$

is well defined and finite for $p < p_c$.

<u>Def</u>: $\xi(p)$ is called the <u>correlation length</u>.

<u>Lemma (Fekete's lemma)</u>.

Let $(u_n)_{n \geq 0}$ be a sequence of numbers in $[-\infty, \infty)$ satisfying

$$\forall m, n \geq 0 \quad u_{m+n} \leq u_m + u_n \qquad \text{"subadditivity"}$$

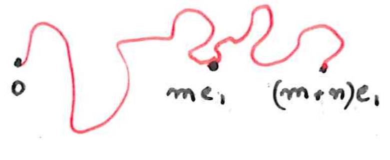
Then the limit of $\left(\frac{u_n}{n}\right)$ exists in $[-\infty, \infty)$ and

$$\lim_{n \to \infty} \frac{u_n}{n} = \inf_{n \geq 0} \left(\frac{u_n}{n}\right).$$

<u>Proof of the theorem.</u>

By FKG inequality, we have $\forall m, n \geq 0$

$$\mathbb{P}[0 \longleftrightarrow (m+n)e_1] \geq \mathbb{P}_p[0 \longleftrightarrow me_1] \times \underbrace{\mathbb{P}_p[me_1 \longleftrightarrow (m+n)e_1]}_{\underset{\substack{\uparrow \\ \text{translation} \\ \text{invariance}}}{= \mathbb{P}[0 \longleftrightarrow ne_1]}}$$



Hence, $u_n = -\log(\mathbb{P}_p[0 \longleftrightarrow n e_1])$ is subadditive and Fekete's lemma concludes that $\left(\frac{u_n}{n}\right)$ converges towards $\inf_{n \geq 0} \left(\frac{u_n}{n}\right)$. ∎

<u>Rk:</u> The definition of $\xi(p)$ can be also rewritten as

$$\mathbb{P}_p[0 \longleftrightarrow n e_1] = e^{-n/\xi(p) + o(n)}.$$

<u>Prop:</u> Let $p < p_c$. $\exists c, C > 0$ s.t. $\forall n \geq 1$

$$\boxed{\frac{1}{C n^{d-1}} e^{-\frac{n}{\xi(p)}} \leq \theta_n(p) \leq C n^{d-1} e^{-n/\xi(p)}}$$

<u>Proof:</u> [Upper bounds] Fix $p < p_c$.

$$\forall n \quad \frac{u_n}{n} \geq \inf_{n \geq 0} \frac{u_n}{n} = \frac{1}{\xi(p)}$$

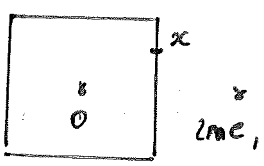
ie $\forall n \quad \mathbb{P}_p[0 \longleftrightarrow n e_1] \leq e^{-n/\xi(p)}$.

By symmetry we can pick $x \in \partial\Lambda_n$ s.t. $x_1 = n$ and

$$\mathbb{P}_p[0 \longleftrightarrow x] = \max_{y \in \partial\Lambda_n} \mathbb{P}_p[0 \longleftrightarrow y].$$

By invariance of $\mathbb{P}_p$ with respect to the reflexion in $\{n\} \times \mathbb{Z}^{d-1}$,

$$\mathbb{P}_p[0 \longleftrightarrow x] = \mathbb{P}_p[x \longleftrightarrow 2n e_1]$$



By FKG inequality

$$\mathbb{P}_p[0 \longleftrightarrow 2me_1] \geq \mathbb{P}_p[0 \longleftrightarrow x]\,\mathbb{P}_p[x \longleftrightarrow 2me_1]$$
$$= \mathbb{P}_p[0 \longleftrightarrow x]^2$$

Finally

$$\theta_m(p) = \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_m]$$
$$\leq \sum_{y \in \partial\Lambda_m} \mathbb{P}_p[0 \longleftrightarrow y]$$
$$\leq |\partial\Lambda_m|\,\mathbb{P}_p[0 \longleftrightarrow x]$$
$$\leq |\partial\Lambda_m|\,\mathbb{P}_p[0 \longleftrightarrow 2me_1]^{\frac{1}{2}}$$
$$\leq C m^{d-1} e^{-m/\xi(p)}.$$

[Lower bound.]

For $1 \leq m \leq n$, we have

$$\theta_{m+n}(p) \overset{\text{indep.}}{\leq} \underbrace{\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_m]}_{=\theta_m(p)}\ \underbrace{\mathbb{P}_p[\partial\Lambda_m \longleftrightarrow \partial\Lambda_{m+n}]}_{\leq \sum_{x \in \partial\Lambda_m} \mathbb{P}_p[x \longleftrightarrow \partial\Lambda_{m+n}]}$$
$$\leq 3^{d-1} m^{d-1}\,\theta_n(p)$$

Set $C = 6^{d-1}$

$$C(m+n)^{d-1}\theta_{m+n}(p) \leq C \times (2n)^{d-1} \times \theta_m(p) \times 3^{d-1} m^{d-1} \theta_n(p)$$
$$\leq \left(C\, m^{d-1}\theta_m(p)\right)\left(C\, n^{d-1}\theta_n(p)\right)$$

Hence the sequence $v_n = \log\left(C n^{d-1}\theta_n(p)\right)$ is subadditive. By Fekete's lemma, we have for every n

$$\frac{v_n}{n} \geq \lim_{n\to\infty} \frac{v_n}{n} = -\frac{1}{\xi(p)}$$

$$\uparrow$$

$$\mathbb{P}_p[0 \leftrightarrow ne_1] \leq \theta_n(p) \leq Cte\, n^{d-1} e^{-n/\xi(p)}$$

Hence, for every $n \geq 1$

$$\theta_n(p) \geq \frac{1}{Cn^{d-1}} e^{-n/\xi(p)}.$$

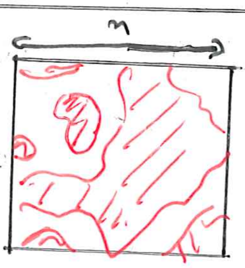
Exercise: Let $p < p_c$. Prove that $\exists c > 0$ s.t.

$$\frac{c\, e^{-\frac{\|x\|_\infty}{\xi(p)}}}{\|x\|_\infty^{4d(d-1)}} \leq \mathbb{P}_p[0 \leftrightarrow x] \leq e^{-\frac{\|x\|_\infty}{\xi(p)}}$$

Rk: More precise estimates, known as Ornstein - Zernike estimates state that $\exists\ c = c(p) > 0$ s.t.

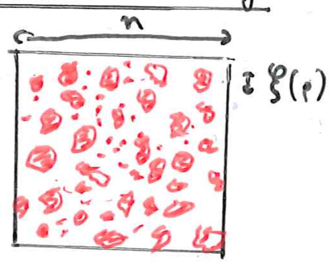
$$\mathbb{P}_p[0 \leftrightarrow ne_1] = \frac{c}{n^{\frac{d-1}{2}}} e^{-(n/\xi(p))} \left(1 + o_{n\to\infty}(1)\right).$$

Geometric intuition concerning the correlation length.



$n \leq \xi(p)$

$\rightarrow$ looks critical.



$n >> \xi(p)$

$\rightarrow$ really looks subcritical.

CHAPTER 2

RENORMALIZATION AND EXPONENTIAL DECAY IN VOLUME.

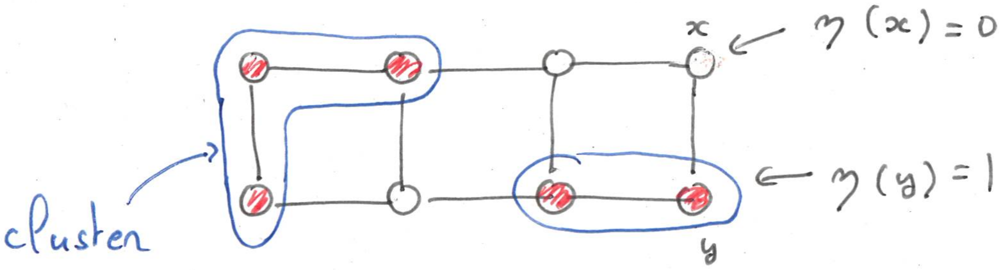
Rk. $\{|C_0| \geq n\} \subset \{0 \leftrightarrow \partial \Lambda_{\frac{1}{3} n^{\frac{1}{d}}}\}$ $\xrightarrow{p<p_c}$ $P_p[|C_0| \geq n] \leq e^{-cn^{\frac{1}{d}}}$ Can we do better?

I PERTURBATIVE REGIME OF SITE PERCOLATION

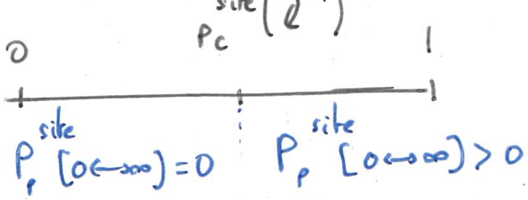
$$G = (\mathbb{Z}^d, E)$$

site percolation:

- configuration : $\eta \in \{0,1\}^{\mathbb{Z}^d}$
- measure : $P_p^{site} = (p\,\delta_1 + (1-p)\,\delta_0)^{\otimes \mathbb{Z}^d}$ (Bernoulli site perco)

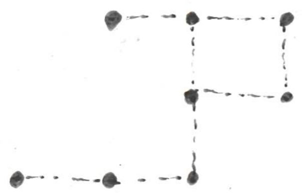


Rk: all what we have done so far extends to Bernoulli site percolation.



Goal: $P_p^{site}[|C_0| \geq n] \leq e^{-cn}$ for p small.

Def: $\boxed{\mathcal{A}_n = \{C \subset \mathbb{Z}^d \text{, connected }, |C| = n\} \quad \text{"animals"}}$



an animal $C \in \mathcal{A}_8$

Not: $\partial^{ext} C = \{y \notin C : \exists x \in C \; x \sim y\}$

Lemma: For every $n \geq 0$ we have

$$\boxed{|\mathcal{A}_n| \leq 4^{dn}}$$

Proof: Let $n \geq 2$. For $C \in \mathcal{A}_n$ we have

$$P_p^{site}[\mathcal{C}_0(\eta) = C] = p^{|C|} \cdot (1-p)^{|\partial^{ext} C|}$$

"cluster of 0"

$$\geq p^{|C|} (1-p)^{(2d-1)|C|}$$

because $|\partial^{ext} C| = \left| \bigcup_{x \in C} \{y \notin C : x \sim y\} \right| \leq (2d-1)|C|$

For $p = \frac{1}{2}$, we have

$$1 \geq P_{\frac{1}{2}}^{site}[|\mathcal{C}_0(\eta)| = n] = \sum_{C \in \mathcal{A}_n} P_{\frac{1}{2}}^{site}[\mathcal{C}_0(\eta) = C]$$

$$\geq |\mathcal{A}_n| \cdot \frac{1}{2^{2dn}} \quad \blacksquare$$

Not. Let $\lambda = \sup_n \{ |\mathcal{A}_n|^{\frac{1}{n}} \} < \infty$

(we have $\forall n \quad |\mathcal{A}_n| \leq \lambda^n$)

Prop. For every $p < \frac{1}{\lambda}$, $\exists c > 0$ s.t.

$$\forall n \geq 1 \quad P_p^{site} [|\mathcal{C}_0| \geq n] \leq e^{-cn}.$$

Proof: $P_p^{site} [|C_0| \geq n] \leq P_p^{site} \left[\bigcup_{C \in \mathcal{A}_n} \{ C \text{ is open} \} \right]$

$$\leq |\mathcal{A}_n| \cdot p^n$$

$$\leq (\lambda p)^n$$

■

Generalization:

Let $(\Omega, \mathbb{P}, \mathcal{F})$ be a probability space, let $k \geq 1$ and $\eta : \Omega \to \{0,1\}^{\mathbb{Z}^d}$ a random configuration s.t.

- η is k-indep. (i.e. $\forall A, B \subset \mathbb{Z}^d$ $d(A,B) \geq k \Rightarrow \eta_{|A}$ and $\eta_{|B}$ indep.)
- η has marginals at most $p_k := \frac{1}{(\lambda e)^{|\Lambda_k|}}$ (i.e. $\forall x \in \mathbb{Z}^d \quad \mathbb{P}[\eta(x) = 1] \leq p_k$)

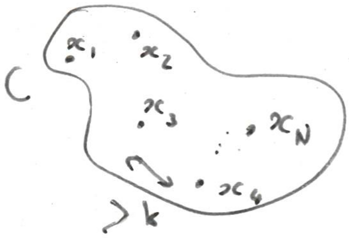
Then $\boxed{\forall n \geq 1 \quad \mathbb{P}[\mathcal{C}_0(\eta) \geq n] \leq e^{-n}}$

Proof: Let $n = |\Lambda_k| \times N$, for some $N \geq 1$.

Let $C \in \mathcal{A}_n$.

There exist $x_1, \dots, x_N \in C$ s.t. $i \neq j \Rightarrow d(x_i, x_j) > k$.

(pick $x_1 \in C$, then $x_2 \in C \setminus \Lambda_k(x_1)$, ...)



$$\mathbb{P}[\forall x \in C \ \eta(x) = 1] \leq \mathbb{P}[\eta(x_1) = \dots = \eta(x_N) = 1]$$

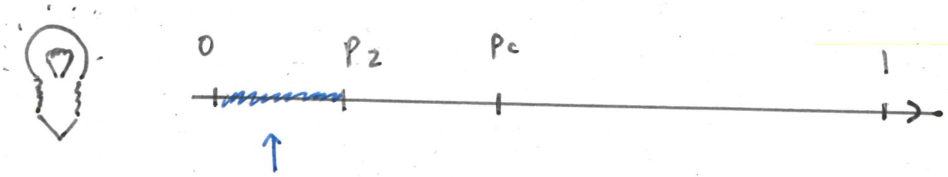
$$\overset{\text{k-indep.}}{\leq} \mathbb{P}[\eta(x_1) = \dots = \eta(x_{N-1}) = 1] \underbrace{\mathbb{P}[\eta(x_N) = 1]}_{\leq p_k}$$

$$\overset{\text{induction}}{\leq} (p_k)^N = \frac{1}{(\lambda e)^n}.$$

Hence
$$\mathbb{P}[|\mathcal{C}_0(\eta)| \geq n] \leq \sum_{C \in \mathcal{A}_n} \mathbb{P}[\forall x \in C \ \eta(x) = 1]$$

$$\leq \lambda^n \times \frac{1}{(\lambda e)^n} = e^{-n} \qquad \blacksquare$$

Exercise: prove that for Bernoulli bond percolation $\forall p < p_2 \ \forall n \ P_p[|C_0| \geq n] \leq e^{-n}$.



Bernoulli bond percolation.

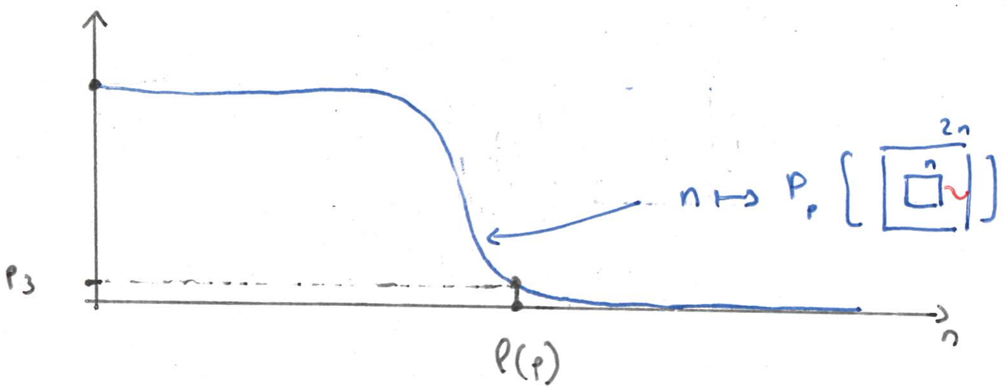
$P_p[|C_0| \geq n] \leq e^{-n}$

2 CORRELATION LENGTH REVISITED

$\mathbb{P}_p$: Bernoulli bond percolation on $(\mathbb{Z}^d, E)$

Def: For $p \in [0,1]$, define

$$\boxed{\ell(p) = \inf\left\{ n : \mathbb{P}_p[\Lambda_n \longleftrightarrow \partial\Lambda_{2n}] \leq p_3 \right\}}$$



Prop. $\boxed{\forall p < p_c \quad \ell(p) < \infty}$

Proof: $\Lambda_n(x) := x + \Lambda_n$. Let $p < p_c$

$$\mathbb{P}_p[\Lambda_n \longleftrightarrow \partial\Lambda_{2n}] \leq \sum_{x \in \partial\Lambda_n} \underbrace{\mathbb{P}_p[x \longleftrightarrow \partial\Lambda_n(x)]}_{=\theta_n(p)}$$

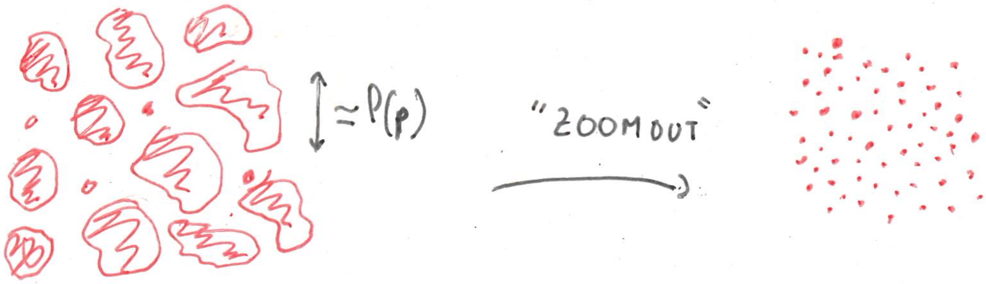
$$\leq |\partial\Lambda_n| e^{-cn} \xrightarrow[n\to\infty]{} 0 \qquad \blacksquare$$

Exercise Prove that $\exists C > 0$ s.t. $\forall p \in [0,1)$

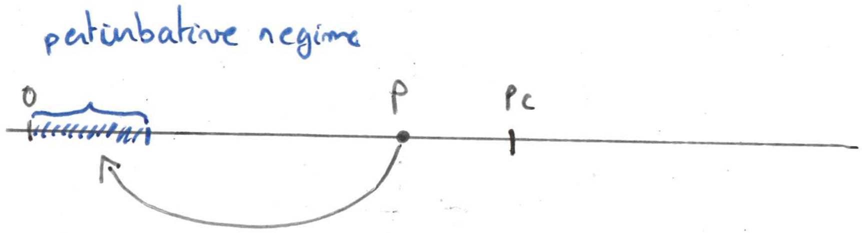
$$\frac{\xi(p)}{2} \leq \ell(p) \leq 1 + C\,\xi(p)\,(2 + \log(\xi(p))).$$

3 EXPONENTIAL DECAY IN VOLUME

idea of renormalization



$p < p_c$ look like $p = \varepsilon$



renormalization.

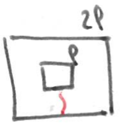
We will use this idea to prove the following thm:

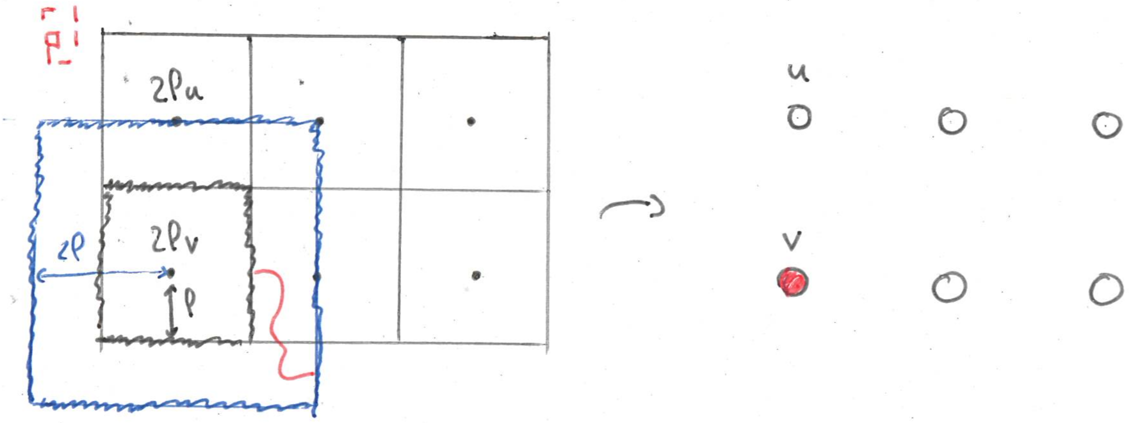
Thm (exponential decay in volume).

> Let $p < p_c$. There exists $c > 0$ s.t.
>
> $$\boxed{\forall n \geq 1 \quad P_p\left[|C_0| \geq n\right] \leq e^{-cn}.}$$

Rk: This theorem implies exponential decay in radius:

$$P_p\left[0 \longleftrightarrow \partial\Lambda_n\right] \leq P_p\left[|C_0| \geq n\right] \leq e^{-cn}.$$

Proof: Let $p < p_c$. Let $\ell = \ell(p)$. $\longrightarrow$ $P_p\Big[$  $\Big] \leq p_3$



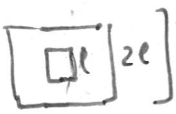
$$\omega \in \{0,1\}^E \longrightarrow \eta \in \{0,1\}^{\mathbb{Z}^d}$$

where $\forall v \in \mathbb{Z}^d$ $\eta(v) := \begin{cases} 1 & \text{if } \Lambda_\ell(2\ell v) \overset{\omega}{\longleftrightarrow} \partial \Lambda_{2\ell}(2\ell v) \\ 0 & \text{otherwise.} \end{cases}$

Observation: $\eta : \Omega = \{0,1\}^E \longrightarrow \{0,1\}^{\mathbb{Z}^d}$

random site percolation (def. on $(\{0,1\}^E, P_p)$)

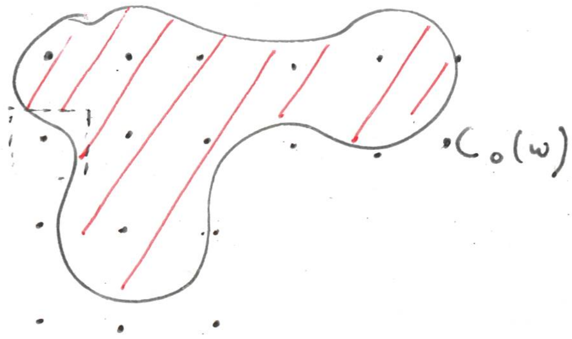
$\rightarrow$ η is 3-indep.

$\rightarrow$ $P_p[\eta(u) = 1] = P_p\Big[$  $\Big] \leq p_3$.

Hence $P_p[\mathcal{C}_0(\eta) \geq k] \leq e^{-ck}$ for every $k \geq 1$

Now, let $n \geq 2|\Lambda_\ell|$.

If $|C_0(\omega)| \geq n$, the $C_0(\omega)$ intersects at least $\left\lfloor \frac{n}{|\Lambda_\ell|} \right\rfloor$ boxes of the form $\Lambda_\ell(2\ell u)$, $u \in \mathbb{Z}^d$.



Hence $P_p\left[\,|C_0(w)| \geq n\,\right] \leq P_p\left[\,\mathscr{C}_0(\eta) \geq \left\lfloor \frac{n}{|\Lambda_\ell|} \right\rfloor\right]$

$$\leq e^{-\left\lfloor \frac{n}{|\Lambda_\ell|} \right\rfloor}.$$

$\blacksquare$

Summary: Step 1 : proof of volume exponential decay in the perturbative regime $p = \varepsilon$.

Step 2 : extension of the result to the whole subcritical regime $p \in [0, p_c)$ via renormalization.

CHAPTER 3 -

UNIQUENESS OF THE INFINITE CLUSTER.

$G = (\mathbb{Z}^d, E)$ $d \geq 2$

For $\omega \in \{0,1\}^E$, write $N(\omega)$ for the number of infinite clusters in the configuration ω.

$\boxed{\text{Thm}: \text{Let } p \in [0,1]. \text{ Either } \mathbb{P}_p[N=0]=1 \text{ or } \mathbb{P}_p[N=1]=1.}$

Exercise: Deduce that

$$N = \begin{cases} 0 & \text{a.s.} \quad \text{if } \theta(p)=0 \\ 1 & \text{a.s} \quad \text{if } \theta(p)>0. \end{cases}$$

1. PROOF OF THE THEOREM.

Let $p \in (0,1)$. By ergodicity $\exists k = k(p) \in \mathbb{N} \cup \{+\infty\}$ s.t. $\mathbb{P}_p[N=k] = 1$.

Lemma 1: $\boxed{k \in \{0, 1, \infty\}.}$

Proof: Assume $1 < k < \infty$.

Let $\mathcal{F}_n = \left\{ \begin{array}{l} \Lambda_n \longleftrightarrow \infty \text{ and all the infinite clusters} \\ \text{intersect the box } \Lambda_n \end{array} \right\}$

For n large enough $\mathbb{P}_p[\mathcal{F}_n] \geq \frac{1}{2}$. (since $1 \leq N < \infty$ a.s.).

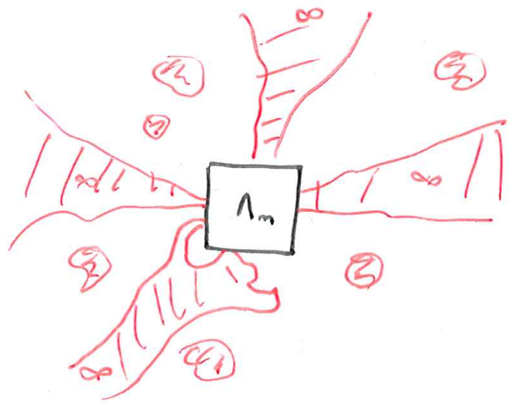


illustration of $\mathcal{F}_n$ for $N=4$.
(Notice that $\mathcal{F}_n$ is independent of the configuration in Λ_n).

$$P_p[N=1] \geq P_p[\mathcal{F}_n \cap \{\text{all the edges in } \Lambda_n \text{ are open}\}]$$

$$\underset{\text{independ.}}{=} P_p[\mathcal{F}_n] \cdot P_p[\text{all the edges in } \Lambda_n \text{ are open}]$$

> 0 Contradiction to $N=k$ a.s. ! ∎

Definition:

Let $\omega \in \{0,1\}^E$. A vertex $x \in \mathbb{Z}^d$ is called a trifurcation (in ω) if
- x has exactly 3 adjacent open edges.
- C_x splits into 3 disjoint infinite clusters if we close the edges adjacent to x.

Not: $T_x = \{x \text{ is a trifurcation}\}$.

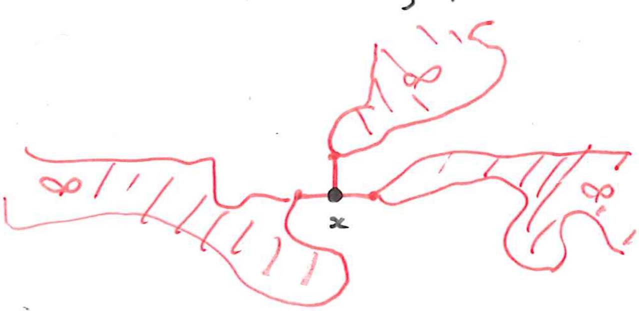


illustration of T_x.

Lemma 2 If $\mathbb{P}_p[N \geq 3] > 0$, then

$$\mathbb{P}_p[T_0] > 0.$$

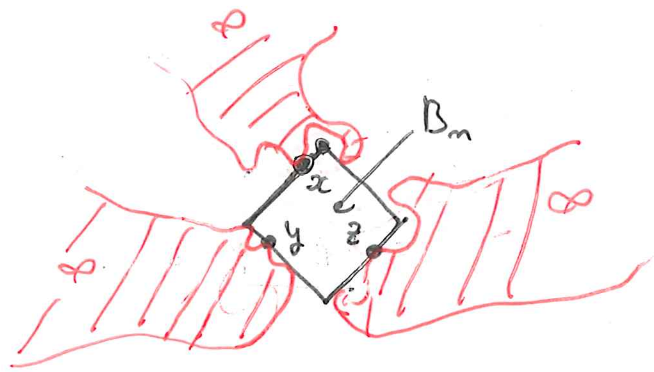
Proof: Let B_n be the ball of radius n centered at 0 for the L^1 distance in $\mathbb{Z}^d$. Pick $n \geq 3$ large enough s.t.

$$\mathbb{P}_p[E_n] > 0$$

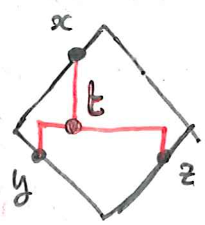
where E_n is the event that at least 3 disjoint infinite clusters intersect B_n. We have

$$0 < \mathbb{P}_p[E_n] \leq \sum_{x,y,z \in \partial B_n} \mathbb{P}_p[E_n(x,y,z)]$$

where $E_n(x,y,z)$ is the event that outside B_n, the clusters of x, y and z are disjoint and infinite.



Let $x,y,z \in \partial B_n$ s.t. $\mathbb{P}_p[E_n(x,y,z)] > 0$.
On can check that there exist a deterministic vertex $t \in B_n \setminus \partial B_n$ and three disjoint paths γ_x, γ_y and γ_z in B_n s.t. γ_i connects t to i for every i



Let $F_n(x,y,z) = \{\gamma_x, \gamma_y, \gamma_z$ are open and all the other edges of B_n are closed$\}$

Proof of the theorem.

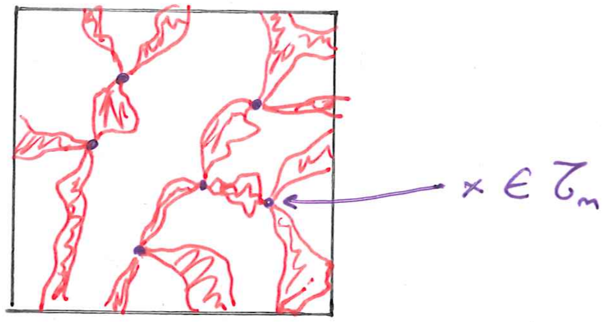
Assume for contradiction that $\mathbb{P}_p[N=\infty]=1$.

By Lemma 2, we have $c := \mathbb{P}[T_0] > 0$.

Define $\mathcal{T}_m(\omega) = \{x \in \Lambda_m : x \text{ is a trifurcation}\}$

By translation invariance, we have

$$\mathbb{E}[|\mathcal{T}_m|] = \sum_{x \in \Lambda_m} \mathbb{P}[T_x] = c \cdot |\Lambda_m|$$



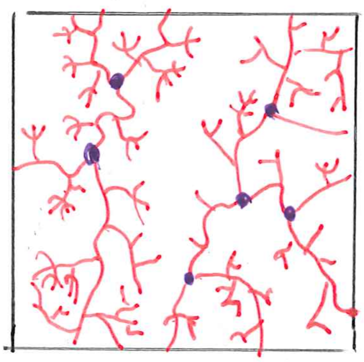
We claim that for every configuration ω, $|\mathcal{T}_m(\omega)| \leq |\partial \Lambda_m|$.

To see this, consider the subgraph of Λ_m obtained by the following peeling procedure.

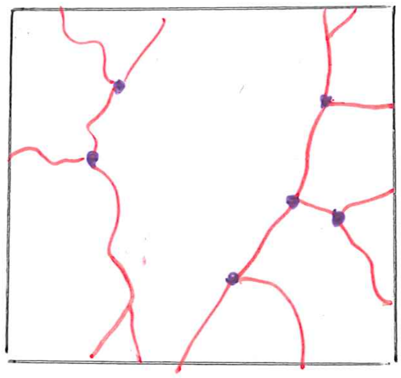
Let $F_0 = \{e_1, \dots, e_n\}$ be the set of open edges in Λ_m

For $i=1,\dots,n$ set $F_i = \begin{cases} F_{i-1} \setminus \{e_i\} & \text{if } e_i \text{ belongs to a cycle of } F_i. \\ F_{i-1} & \text{otherwise} \end{cases}$

After this first step the graph induced by F_n is a forest.



Then, remove all the vertices of degree 1 in the graph induced by F_n, except the vertices on $\partial \Lambda_n$. Repeat this operation until the time there is no more vertex of degree 1, except those on $\partial \Lambda_n$. Consider the graph induced by the remaining edges.



Write N_1 for the vertices of degree 1 in this graph, and

$N_{\geq 3}$ ______________________ ≥ 3 __________ .

Notice that $N_1 \leq |\partial \Lambda_n|$ and $N_{\geq 3} \geq |\mathcal{T}_n|$ (because the trifurcations have not been deleted during the "peeling" procedure. By applying Lemma 3 to each of the connected component of the graph above, we obtain

$$|\mathcal{T}_n| \leq N_{\geq 3} \leq N_1 \leq |\partial \Lambda_n|$$

Taking the expectation, we obtain

$$c\,|\Lambda_n| \leq |\partial \Lambda_n|$$

which is a contradiction to $\dfrac{|\partial \Lambda_n|}{|\Lambda_n|} \xrightarrow[n \to \infty]{} 0$ ∎

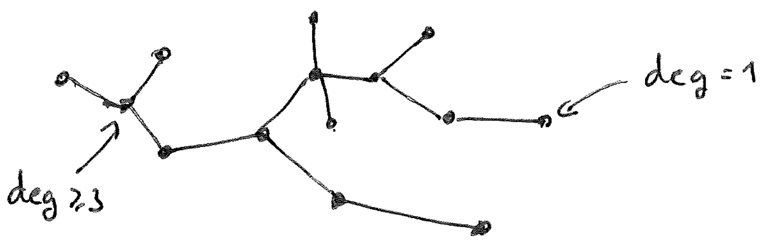
Finally by independence.

$$0 < \mathbb{P}_p[E_m(x,y,z)]\,\mathbb{P}_p[F_m(x,y,z)]$$
$$\leq \mathbb{P}_p[E_m(x,y,z) \cap F_m(x,y,z)]$$
$$\leq \mathbb{P}_p[T_t] = \mathbb{P}_p[T_0]$$

↑ translation invariance.

Lemma 3. Let (T, F) be a finite tree. (a finite connected graph with no cycle.)
Let $N_1 = |\{x \in T : \deg(x) = 1\}|$, $N_{\geq 3} = |\{x \in T : \deg(x) \geq 3\}|$

Then
$$N_1 \geq 2 + N_{\geq 3}.$$



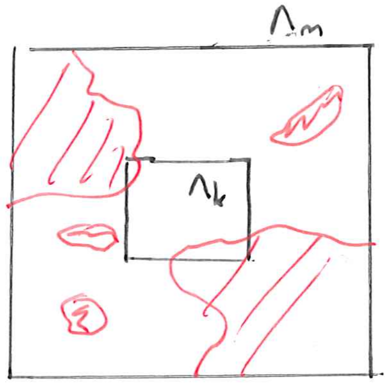
a tree with $N_1 = 7$, $N_{\geq 3} = 4$.

Proof: We have $|T| = |F| + 1$ (by induction on $|T|$).
Write $N_2 = |\{x \in T : \deg(x) = 2\}|$
By counting the edges of the tree in two different ways, we find $2|F| = \sum_{x \in T} \deg(x) \geq N_1 + 2N_2 + 3N_{\geq 3}$.

Since $2|F| = 2|T| + 2 = 2(N_1 + N_2 + N_{\geq 3}) + 2$,
we obtain $N_1 \geq N_{\geq 3} + 2$ ∎

2 UNIQUENESS ZONE.

For $1 \leq k \leq n < \infty$, let



Above, $k=2$

$$\boxed{U_{k,n} = \{ K \leq 1 \}}$$

where K counts the number of disjoint clusters in Λ_n intersecting Λ_k and $\partial \Lambda_n$.

Rk: • $U_{k,n}$ is neither an increasing event nor a decreasing one.

• $\mathbb{P}_p[U_{k,n}]$ is increasing in n, decreasing in k.

Prop. For every $\varepsilon > 0$ and $k \geq 1$, $\exists$ $n = n(\varepsilon, k)$ s.t.

$$\forall p \in [0,1] \quad \mathbb{P}_p[U_{k,n}] > 1 - \varepsilon.$$

Proof: Fix $\varepsilon > 0$ and $k \geq 1$. Define

$$O_m = \{ p \in [0,1] \text{ s.t. } \mathbb{P}_p[U_{k,m}] > 1-\varepsilon \}. \quad \text{(open)}$$

By uniqueness of the infinite cluster (when it exists), we have $\forall p \in [0,1]$ $\exists$ $m(p) \geq k$ s.t. $\mathbb{P}_p[U_{k,m(p)}] > 1 - \varepsilon$.

Hence $[0,1] = \bigcup_{m \geq k} O_m \underset{\text{compactness}}{=} \bigcup_{1 \leq i \leq i_0} O_{m_i}$.

Choosing $n = \max_{1 \leq i \leq i_0} m_i$ concludes the proof. ■

CHAPTER 4:

PERCOLATION ON $\mathbb{Z}^2$

In this chapter, we fix $d=2$. The graph $(\mathbb{Z}^2, E)$ is planar, which provides several useful tools for the study of percolation: planar graphs satisfy duality relations, that will have deep consequences for percolation, also, it will be easy to "force" open paths to intersect.

1.) $p_c = \frac{1}{2}$

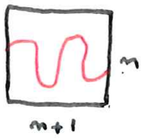
Thm [Kesten '80]

$$\boxed{p_c = \frac{1}{2} \text{ and } \theta(p_c) = 0}$$

Duality for planar percolation.

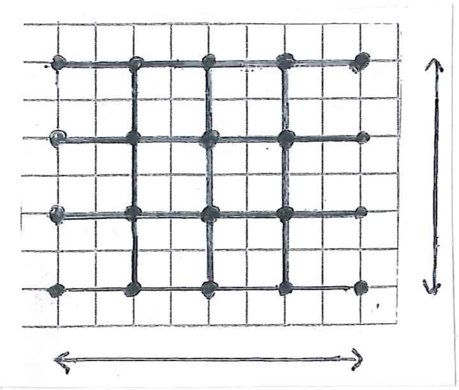
	primal	dual.
graphs.	$(\mathbb{Z}^2, E)$	$((\mathbb{Z}^2)^*, E^*) = (\mathbb{Z}^2, E)$ translated by $(\frac{1}{2}, \frac{1}{2})$.
percolation	$\omega \in \{0,1\}^E$	$\omega^* \in \{0,1\}^{E^*}$ $(\omega^*(e^*) = 1 - \omega(e))$
	$\omega \sim \mathbb{P}_p$	$\omega^* \sim \mathbb{P}_{1-p}$.

Lemma: For $p = \frac{1}{2}$

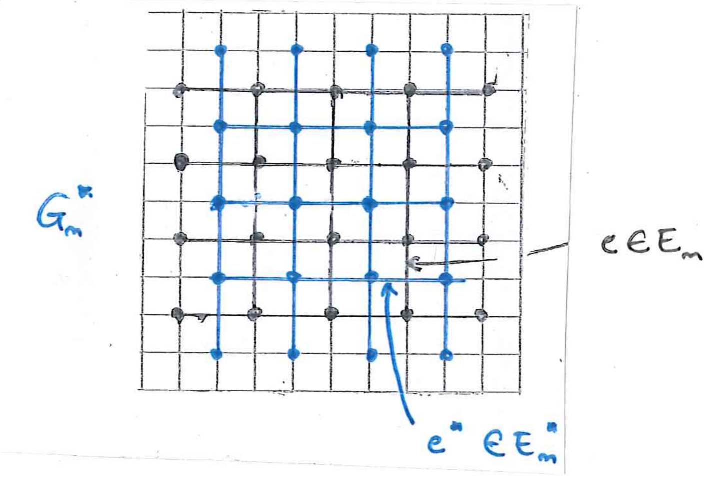
$$\forall n \geq 0 \quad \mathbb{P}_p\Big[$$

$$\Big] = \frac{1}{2}$$

Proof:

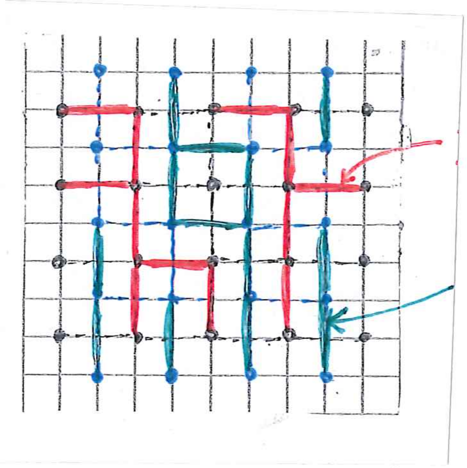
For $m \geq 1$, consider the graph $G_m = (V_m, E_m)$ defined by

$G_m =$ 

Let $G_m^* = (V_m^*, E_m^*)$ be the blue graph below



Notice that
- an edge e of E_m crosses exactly one edge $e^* \in E_m^*$
- G_m^* is a rotated and translated version of G_m.

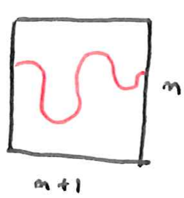


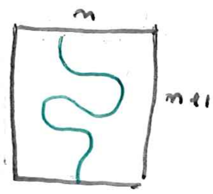
Admitted combinatorial result:

$$(\exists \text{ left-right open path in } \omega) \iff (\exists \text{ no top-down path in } \omega^*)$$

For a proof of this result, see e.g. [Bollobás, Riordan, Chap. 3]

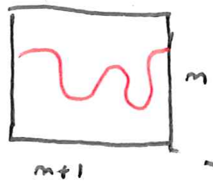
This implies

$$\mathbb{P}_p\Big[$$ 

$$\Big] + \mathbb{P}_p\Big[$$ 

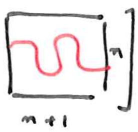
$$\Big] = 1$$

If $p = \frac{1}{2}$, the two probabilities are equal and we have

$$\forall m \geq 0 \quad \mathbb{P}_{\frac{1}{2}}\Big[$$ 

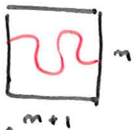
$$\Big] = \frac{1}{2}.$$

Proof of the theorem:

For $p = \frac{1}{2}$ we do not have $\mathbb{P}_p\Big[$ 

$\Big] \xrightarrow[n\to\infty]{} 1$.

Hence $\theta(\frac{1}{2}) = 0$ and $p_c \geq \frac{1}{2}$.

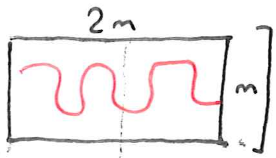
For $p = \frac{1}{2}$ we do not have $\mathbb{P}_p\Big[$ 

$\Big] \xrightarrow[m\to\infty]{} 0$.

(In particular there is not exponential decay of the connection probabilities.

Hence $p_c \leq \frac{1}{2}$. ■

Question:

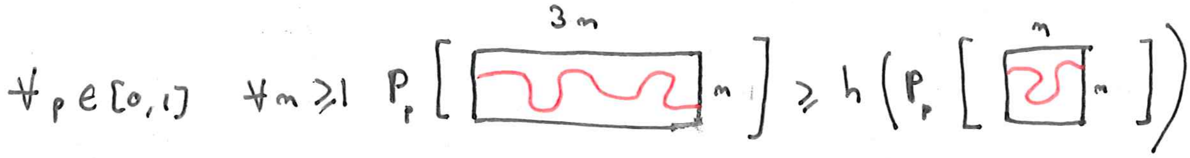
For $p = \frac{1}{2}$, do we have $\inf_{m \geq 0} \mathbb{P}_p\Big[$ 

$\Big] > 0$?

2. Russo-Seymour-Welsh Theorem.

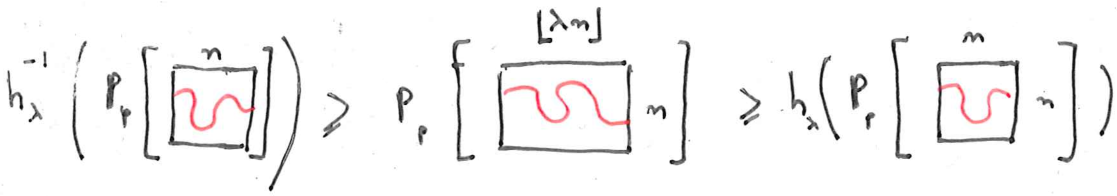
Thm [RSW '78]

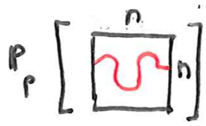
There exists $h : [0,1] \longrightarrow [0,1]$ continuous, (strictly) increasing s.t. $h(0)=0$, $h(1)=1$ and



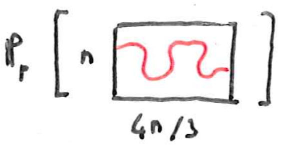
Exercise:

Let $\lambda > 0$. Prove that there exists $h_\lambda : [0,1] \to [0,1]$ as above such that $\forall p \in [0,1]$ $\forall n \geq 1/\lambda$

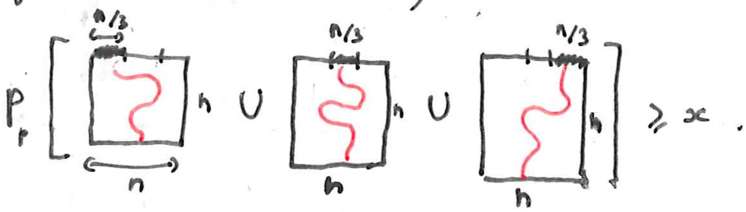


Proof: Fix $p \in [0,1]$, $n \geq 1$. Set $x =$ 

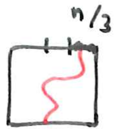
Assume for simplicity that $n \in 6\mathbb{N}$.

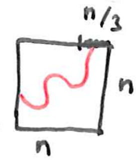
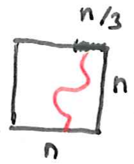
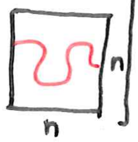
Step 1:  $\geq g(x)$ where $g(x) := x^2\left(1-(1-x)^{1/6}\right)^2$.

First, by rotation invariance,



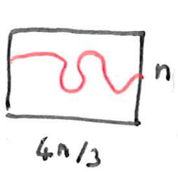
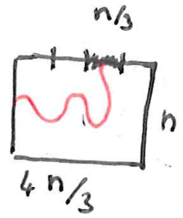
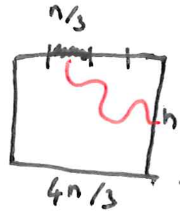
Hence, by reflection invariance and the square-root trick,

<u>case 1:</u> $\mathbb{P}_p\Big[$  $\Big] \geq 1-(1-x)^{\frac{1}{3}}$.

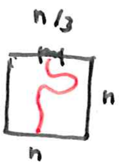
Then $\mathbb{P}_p\Big[$  $\Big] \geq \mathbb{P}_p\Big[$  $\cap$  $\Big]$

$$\overset{FKG}{\geq} x\left(1-(1-x)^{\frac{1}{3}}\right) \overset{\text{trivial}}{\geq} x\left(1-(1-x)^{\frac{1}{6}}\right),$$

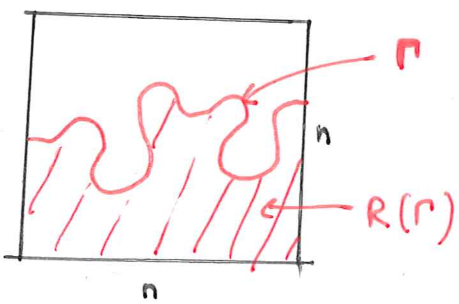
and

$\mathbb{P}_p\Big[$  $\Big] \geq \mathbb{P}_p\Big[$  $\cap$  $\Big]$

$$\overset{FKG}{\geq} x^2\left(1-(1-x)^{1/6}\right)^2.$$

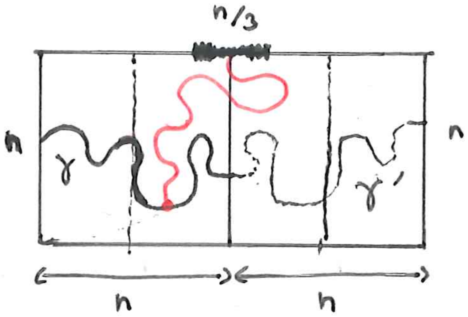
<u>Case 2:</u> $\mathbb{P}_p\Big[$  $\Big] \geq 1-(1-x)^{1/3}$.

If there exists a left-right open path in $[0,n]^2$, define Γ to be the lowest left-right open path in $[0,n]^2$. Set $\Gamma = \emptyset$ if there is no such path.

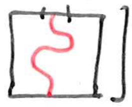


<u>To check:</u> Γ is well defined, and for every $\gamma \neq \emptyset$ admissible, the event $\{\Gamma = \gamma\}$ is measurable w.r.t the configuration in the region $R(\gamma)$ below γ.

Fix $\gamma \neq \emptyset$ a left-right admissible path. Let γ' be the image of γ by the reflection in the line $\{n\} \times \mathbb{Z}$.

$E_\gamma =$ 

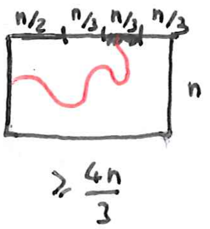
Let E_γ (resp. $E_{\gamma'}$) be the event that $[\frac{5n}{6}, \frac{7n}{6}] \times \{n\}$ is connected to γ (resp. to γ') in the region of $[\frac{n}{2}, \frac{3n}{2}] \times [0,n]$ above $\gamma \cup \gamma'$.

Notice that $\mathbb{P}_p[E_\gamma \cup E_{\gamma'}] \geq \mathbb{P}[$  $] \geq 1-(1-x)^{1/3}$.

Since $\mathbb{P}_p[E_\gamma] = \mathbb{P}_p[E_{\gamma'}]$, the square-root trick implies.

$$\mathbb{P}_p[E_\gamma] \geq 1-(1-x)^{1/6}.$$

Notice that $\{\Gamma = \gamma\}$ and E_γ are independent. Hence

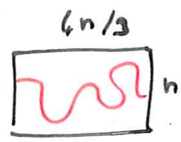
$\mathbb{P}[$  $]$

$$\geq \sum_{\gamma \neq \emptyset} \mathbb{P}_p[\{\Gamma = \gamma\} \cap E_\gamma]$$

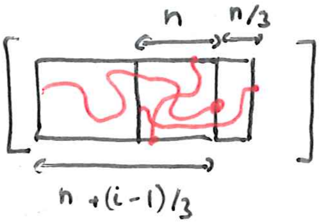
$$\overset{\text{indep.}}{=} \sum_{\gamma \neq \emptyset} \mathbb{P}_p[\Gamma = \gamma]\, \mathbb{P}_p[E_\gamma]$$

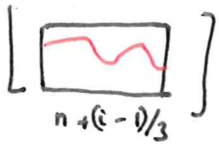
$$\geq \underbrace{\sum_{\gamma \neq \emptyset} \mathbb{P}_p[\Gamma = \gamma]}\, \left(1-(1-x)^{1/6}\right).$$

$= \mathbb{P}_p[$  $] = x$

We conclude as in the first case that $\mathbb{P}_p[$  $] \geq x^2\left(1-(1-x)^{1/6}\right)^2$.

Step 2: Iteration

$\forall i \geq 2$ $\quad P_p$  $\geq P$ 

FKG $\geq P$  $\times\, x \times g(x)$

induction $\geq g(x) \cdot [x\, g(x)]^{i-1}$.

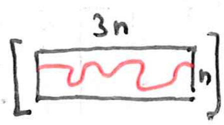
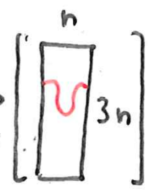
This concludes the proof by setting $h(x) = g(x)\,[x\, g(x)]^5$. ∎

3 CRITICAL BEHAVIOUR.

In this section, we fix $p = p_c = \frac{1}{2}$, and write $P = P_{\frac{1}{2}}$.

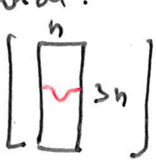
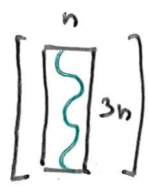
Thm (Box-crossing property).

For $p = p_c$, there exists $c > 0$ s.t.

$\forall n \geq 1$ $\qquad c \leq P$  $\leq P$  $\leq 1 - c$.

Proof: First inequality: use P  $= \frac{1}{2}$ + RSW.

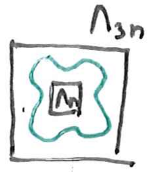
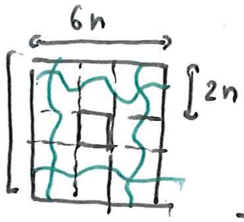
Second inequality: trivial.

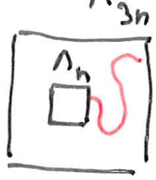
Third inequality: P  $= 1 - P$  $\leq 1 - c$.

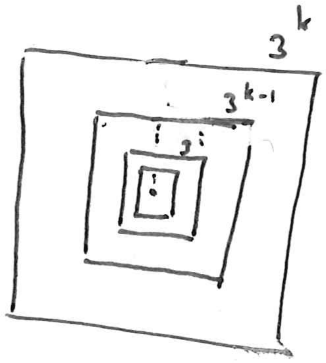
duality $\qquad$ first inequality.

Corollary 1: [Polynomial bound on the 1-arm probability.]

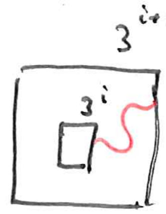
$\exists c > 0$ s.t. $\forall n \geq 1$ $\quad \theta_n\left(\tfrac{1}{2}\right) \leqslant \dfrac{1}{n^c}$.

Pf: $\mathbb{P}\Big[$  $\Big] \geqslant \mathbb{P}\Big[$  $\Big] \overset{\text{FKG}}{\geqslant} c^4 =: c_0$

Hence $\mathbb{P}\Big[$  $\Big] = 1 - \mathbb{P}_p\Big[$  $\Big] \leqslant 1 - c_0$



By independence

$$\theta_{3^k}\left(\tfrac{1}{2}\right) \leqslant \prod_{0 \leq i < k} \mathbb{P}\Big[$$  $$\Big] \leqslant (1 - c_0)^k$$

Choosing $k = \lfloor \log_3 n \rfloor$ for $n \geq 3$ concludes the proof.

Conjecture:

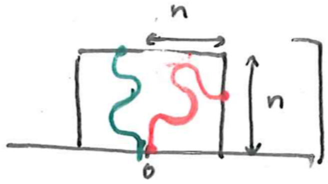
$\exists c > 0$ s.t. $\quad \theta_n(p) \underset{n\to\infty}{\sim} \dfrac{c}{n^{5/48}}$

Rk: the exponent $5/48$ has been proved for site percolation on the triangular lattice.

Corollary 2: (A universal arm exponent)

$\exists c > 0$ s.t $\forall n \geq 1$

$$\frac{c}{n} \leq P\Big[$$



$$\Big] \leq \frac{1}{c} \cdot \frac{1}{n}$$

Proof: exercise.

4 SUPERCRITICAL PERCOLATION.

Key remark: $p > p_c(\mathbb{Z}^2) \iff 1-p < p_c((\mathbb{Z}^2)^*)$

Hence $\forall p > p_c$ $\forall x \in (\mathbb{Z}^2)^*$

$$\forall n \geq 1 \quad P_p\Big[$$

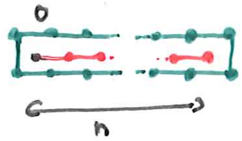
$$\Big] \leq e^{-cn}.$$

Thm [exponential decay of the radius of a finite cluster]

Let $p_c < p < 1$. There exist $c_0, c_1 > 0$ s.t.

$$\forall n \geq 1 \qquad e^{-c_0 n} \leq P_p[0 \longleftrightarrow \partial\Lambda_n,\ 0 \not\longleftrightarrow \infty] \leq e^{-c_1 n}.$$

Proof: Lower bound

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n,\ 0 \not\longleftrightarrow \infty] \geq \mathbb{P}_p\Big[$$  $$\Big]$$

$$= p^n (1-p)^{2n+4}$$

Upper bound.

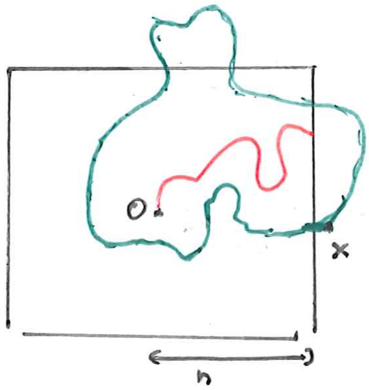


Fig: the event $\{0 \longleftrightarrow \partial\Lambda_n,\ 0 \not\longleftrightarrow \infty\}$.

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n,\ 0 \longleftrightarrow \infty] \leq \mathbb{P}_p\left[\begin{matrix}\exists \text{ dual open circuit with diameter} \\ \text{longer than } n,\ \text{intersecting } \partial\Lambda_n\end{matrix}\right]$$

$$\leq \mathbb{P}_p\left[\exists x \in (\mathbb{Z}^2)^* \text{ s.t. } \|x\|_\infty \geq n,\ x \longleftrightarrow \partial\Lambda_{\|x\|}(x)\right]$$

$$\leq C n e^{-cn} \leq e^{-\frac{c}{2}n} \quad \text{for } n \text{ large enough.} \blacksquare$$

Exercise: (Supercritical correlation length.)

Let $p > p_c$. Using duality, prove that

$$\xi(p) = \lim_{n\to\infty}\left(-\frac{\log\left(\mathbb{P}_p[0 \longleftrightarrow n e_1,\ 0 \not\longleftrightarrow \infty]\right)}{n}\right)^{-1}$$

and show that $\xi(p) = 2\,\xi(1-p)$

$\uparrow$

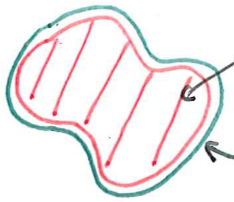
"subcritical correlation length"

Thm [stretch exponential decay in volume]

Let $p_c < p < 1$. There exist $c_0, c_1 > 0$ s.t.

$$\forall n \geq 1 \qquad e^{-c_0\sqrt{n}} \leq \mathbb{P}_p[\,|C_0| \geq n,\, 0 \not\leftrightarrow \infty\,] \leq e^{-c_1\sqrt{n}}.$$

idea:



Proof: upper bound.

If $|C_0| \geq n$ then $0 \leftrightarrow \partial\Lambda_{\frac{\sqrt{n}}{3}}$. Hence,

$$\mathbb{P}_p[\,|C_0| \geq n,\, 0 \not\leftrightarrow \infty\,] \leq \mathbb{P}_p[\,0 \leftrightarrow \partial\Lambda_{\frac{\sqrt{n}}{3}},\, 0 \not\leftrightarrow \infty\,]$$
$$\leq e^{-c\sqrt{n}/3}.$$

lower bound. $\theta = \theta(p)$

Let $k = \left\lceil \sqrt{\frac{n}{\theta}} \right\rceil$ Define $N = \sum_{x \in \Lambda_k} \mathbb{1}_{x \leftrightarrow \partial\Lambda_k}$.

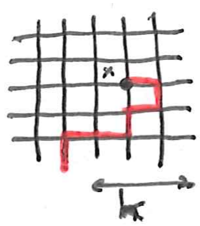


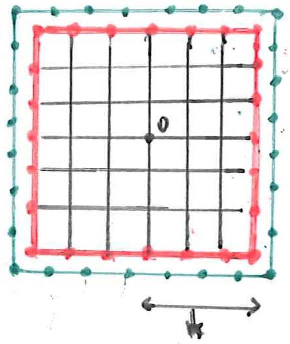
Fig: the event $x \leftrightarrow \partial\Lambda_k$

We have $\mathbb{E}_p[N] = \sum_{x \in \Lambda_k} \mathbb{P}_p[x \leftrightarrow \partial\Lambda_k] \geq |\Lambda_k| \cdot \theta$

Hence, by Markov inequality $\mathbb{P}_p[N \geq \frac{\theta}{2}|\Lambda_k|] \geq \frac{\theta}{2}$.

(indeed $\mathbb{P}_p[\,|\Lambda_k| - N \geq (1-\frac{\theta}{2})|\Lambda_k|\,] \leq \frac{1-\theta}{1-\theta/2} = 1 - \frac{\theta/2}{1-\theta/2} \leq 1-\theta/2$)

Now, let F be the event that all the edges with both extremities in $\partial\Lambda_k$ are open and all the edges of $\Delta\Lambda_k$ are closed.

F = 

$$\mathbb{P}_p[F] \geq [p(1-p)]^{|\Delta\Lambda_k|} \geq e^{-c_0 k}.$$

$$\mathbb{P}_p\left[|C_0| \geq n\right] \geq \mathbb{P}_p\left[|C_0| \geq \frac{\theta}{2}|\Lambda_k|\right]$$

$$\geq \mathbb{P}_p\left[\{0 \longleftrightarrow \partial\Lambda_k, N \geq \frac{\theta}{2}|\Lambda_k|\} \cap F\right]$$

$$\overset{\text{indep.}}{=} \mathbb{P}_p\left[0 \longleftrightarrow \partial\Lambda_k, N \geq \frac{\theta}{2}|\Lambda_k|\right] \mathbb{P}_p[F]$$

$$\overset{\text{FKG}}{\geq} \frac{\theta^2}{2} e^{-c_0 k}.$$

■