Mathematical Finance

Exercise sheet 1

Exercise 1.1 Let $(M_n)_{n\in\mathbb{N}}$ be a martingale such that $M_0 = 0$ and

$$|M_n - M_{n-1}| \leq a_n \quad P\text{-a.s.}$$

for each n and a sequence (a_n) of non-negative constants, with $\sum_{i=1}^{\infty} a_i^2 = A^2 < \infty$.

(a) Prove that M is bounded in L^2. Deduce that $M_n \to M_\infty$ almost surely and in L^2, for some M_∞ in L^2.

(b) Show that

$$P\left(\sup_{k\geq 0} M_k \geq c\right) \leq \exp\left(-\frac{c^2}{2A^2}\right),$$

for any $c > 0$.

Hint: Try applying Doob's maximal inequality to $(e^{\lambda M_n})$, for some $\lambda > 0$. You may use the inequality $\cosh(x) \leq e^{x^2/2}$ (for $x \in \mathbb{R}$).

Exercise 1.2 Let μ be a probability measure on $(0, +\infty)$. Consider (on some probability space) independent $N, Y_1, Y_2, Y_3, ...$ where each Y_i has distribution μ and $N = (N_t)_{t\in[0,1]}$ is a Poisson process on $[0,1]$ of rate $\lambda > 0$. Consider the compound Poisson process X on $[0,1]$ given by

$$X_t = \sum_{i=1}^{N_t} Y_i.$$

(a) Find a necessary and sufficient condition for X to be a submartingale with respect to its natural filtration.

(b) Show that under that condition, X is a submartingale of class (D). Find a decomposition

$$X_t = M_t + A_t \quad \forall t \in [0,1],$$

where M is a martingale and A is an increasing predictable process, both with càdlàg trajectories.

(c) Show through direct calculations that X is a good integrator.

Exercise 1.3 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis satisfying the usual conditions. Let M be a local martingale and X an adapted càdlàg process. Prove the following statements:

(a) M is a uniformly integrable martingale if and only if it is of class (D).

(b) Let X have terminal value X_∞. Then X is a uniformly integrable martingale if and only if for all stopping times τ, the variable X_τ is integrable and $E[X_\tau] = E[X_0]$.

(c) Let X be predictable and τ a stopping time. Then $X_\tau \mathbb{1}_{\tau<\infty}$ is $\mathcal{F}_{\tau-}$-measurable.

(d) Let τ be a predictable finite stopping time. Then $M_{\tau-} = E[M_\tau \mid \mathcal{F}_{\tau-}]$.

(e) Let M be predictable. Then M is continuous.

Hint: For (d), you may use the fact that τ is a predictable stopping time if and only if there exists an *announcing sequence* $(\tau_n)_{n=1}^\infty$ for τ, defined as an increasing sequence of stopping times such that $\tau_n \uparrow \tau$ and $\tau_n < \tau$ P-a.s. on $\tau > 0$.

Exercise 1.4 Let A be an increasing locally integrable process with $A_0 = 0$. Show that, for an increasing predictable process A^p with $A_0^p = 0$, the following conditions are equivalent:

1. $A - A^p$ is a local martingale;
2. $E[A_\tau^p] = E[A_\tau]$ for all stopping times τ;
3. $E[(H \bullet A^p)_\infty] = E[(H \bullet A)_\infty]$ for all nonnegative simple predictable processes H.

Show that there exists a unique such process A^p, known as the *dual predictable projection* or *compensator* of A.

Hint: Use the Doob-Meyer decomposition theorem to prove uniqueness and existence.