Mathematical Finance

Exercise sheet 10

Exercise 10.1 Consider the Black-Scholes market model with a riskless asset $\tilde{S}^0$ and a d-dimensional risky asset $\tilde{S}$ with dynamics defined by the following equations:

$$\begin{aligned} d\tilde{S}_t^0 &= \tilde{S}_t^0 r dt, \\ d\tilde{S}_t &= \operatorname{diag}(\tilde{S}_t)(\mu dt + \sigma dB_t), \end{aligned}$$

where we take constant $r > 0$, $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d \times d}$ an invertible matrix, while B is a d-dimensional Brownian motion.

(a) Write down an expression for the (undiscounted) value of a self-financing portfolio with initial wealth 1 that trades in the risky asset according to a strategy ϑ.

(b) Consider the set $\mathcal{X}$ of wealth processes generated by self-financing portfolios with initial wealth 1 whose value is bounded below by 0. Compute the numéraire process for $\mathcal{X}$.

Exercise 10.2 Consider one riskless asset of constant price and one risky asset satisfying the SABR model:

$$dS_t = \sigma_t S_t dW_t,$$

where σ, the stochastic volatility, itself satisfies the equation

$$d\sigma_t = \alpha \sigma_t dB_t.$$

We take $\alpha, S_0, \sigma_0 > 0$ and W, B to be two independent Brownian motions.

Compute the superreplication price of European call and put options, $(S_T - K)^+$ and $(K - S_T)^+$, for $K \in \mathbb{R}$.

Exercise 10.3 Consider the model

$$dS_t = \sigma(t, S_t) S_t dW_t, \quad S_0 = s_0 > 0,$$

for some $C^{1,2}$ positive function σ, and assume that there exists a $C^{1,2}$ function f such that $f(t, \cdot)$ is the density of S_t for each $t \geq 0$. Show that

$$\sigma(T, K) = \frac{1}{K} \sqrt{\frac{2 \frac{\partial C}{\partial T}(T, K)}{\frac{\partial^2 C}{\partial K^2}(T, K)}},$$

where $C(t, K) = E[(S_t - K)^+]$ for $K > 0$.
Hint: Consider the value process of some payoff $h(S_T)$, for h smooth enough.

Exercise 10.4 Let

$$dY_t = \kappa(\theta - Y_t)dt + \beta \sqrt{Y_t} dW_t, \quad Y_0 = y_0 > 0, \tag{1}$$

where W is a Brownian motion, $\kappa, \theta, \beta > 0$ are constants satisfying the Feller condition $2\kappa\theta > \beta^2$. Show that

$$E\left[\frac{1}{T} \int_0^T Y_t dt\right] = \frac{1 - e^{-\kappa T}}{\kappa T} Y_0 + \left(1 - \frac{1 - e^{-\kappa T}}{\kappa T}\right) \theta. \tag{2}$$

Exercise 10.5 (Python) Compute the expectation (2) by simulating the paths of (1).