Mathematical Finance

Exercise sheet 12

Exercise 12.1 Let U be a utility function satisfying the Inada conditions, i.e. $U \in C^1(\mathbb{R}_+;\mathbb{R})$ is strictly increasing, strictly concave and

$$U'(0) := \lim_{x\searrow 0} U'(x) = +\infty$$
$$U'(+\infty) := \lim_{x\to+\infty} U'(x) = 0.$$

Let J be the Legendre transform of $-U(-\cdot)$,

$$J(y) := \sup_{x>0}(U(x) - xy),$$

and denote by $I := (U')^{-1}$ the inverse of the derivative of U.

Show the following properties:

1. J is strictly decreasing and strictly convex.
2. $J'(0) = -\infty$, $J'(+\infty) = 0$, $J(0) = U(+\infty)$ and $J(+\infty) = U(0)$.
3. For any $x > 0$,
$$U(x) = \inf_{y>0}(J(y) + xy).$$
4. For any $y > 0$,
$$J(y) = U(I(y)) - yI(y).$$
5. $J' = -I$.

Exercise 12.2 Let the financial market $S = (S_k)_{k=0,\dots,N}$ be defined over the *finite* filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0,\dots,N}, P)$ and satisfy $\mathcal{M}^a(S) \neq \emptyset$, and let U be a utility function satisfying the Inada conditions. Consider the value functions

$$u(x) = \sup_{X_T\in C(x)} E[U(X_T)] \text{ and } v(y) = \inf_{Q\in\mathcal{M}^a(S)} E\left[V\left(y\frac{dQ}{dP}\right)\right],$$

where V is the convex conjugate of U and

$$C(x) = \{X_T \in L^0(\Omega, \mathcal{F}_T, P) \mid \forall Q \in \mathcal{M}^a(S) : E_Q[X_T] \le x\}.$$

Show that the optimisers $\hat{X}_T(x)$, $\hat{Q}(x)$ and $\hat{y}(x)$ satisfy $U'\left(\hat{X}_T(x)\right) = \hat{y}(x)\frac{d\hat{Q}(x)}{dP}$ for each $x \in \mathrm{dom}(U)$.

Exercise 12.3 (optional) Consider the utility function $u_\gamma(x) = \frac{x^\gamma}{\gamma}$, for $x > 0$ and $\gamma \in (-\infty, 1)\setminus\{0\}$. Show that $u_\gamma(x) - \frac{1}{\gamma} \to \log x$ as $\gamma \to 0$. Compute the conjugate functions of u_γ and log.

Exercise 12.4 Assume that the interest rate is 0, i.e. there exists a riskless asset with constant value 1, and consider the Bachelier model

$$dS_t = \mu dt + \sigma dB_t, \quad S_0 \in \mathbb{R},$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$.

Compute the optimal utility and optimal strategy associated with the problem

$$J_0 = \sup_{\vartheta \in \Theta^x_{\mathrm{adm}}} E\left[u\left(x + \int_0^T \vartheta_s dS_s\right)\right],$$

for the cases of power utility $u_\gamma(x)$ and log-utility $u(x) = \log(x)$.

Hint. To find a good ansatz for the log-utility case, try (heuristically) taking a limit of the power utility case as $\gamma \to 0$.