Mathematical Finance

Exercise sheet 13

Exercise 13.1 Consider a complete financial market with time interval $[0, T]$, riskless asset $S_t^0 = e^{rt}$ for some $r \in \mathbb{R}$ and a d-dimensional risky asset S, with their natural filtration and unique separating measure Q.

(a) Find the arbitrage-free price at time t of a bounded European ($\mathcal{F}_T$-measurable) payoff H, denoted by $\pi_t^E(H)$.

(b) Let $(U_t)_{t \in [0,T]}$ be a non-negative bounded adapted process. Find the arbitrage-free price at time t of the American option with payoff U, denoted by $\pi_t^A(U)$.

(c) Give an alternative characterisation of $\pi_t^A(U)$ as a Snell envelope.

(d) In terms of an European option, give a necessary and sufficient condition for a given stopping time τ to be an optimal exercise time of the American option.

(e) Suppose that $r \geq 0$ and that the riskless asset follows a Black-Scholes model. Show that the American call option has the same value as the European call option.

(f) Suppose that U is continuous and uniformly bounded. Define the stopping time $\tau = \inf\{t \geq 0 : \pi_t^A(U) = U_t\}$. Show that $\tau \leq T$, that τ is an optimal exercise time for the American option and that the stopped process $(\pi^A/S^0)^\tau$ is a Q-martingale.

(g) Suppose $r = 0$. Let M be a non-negative local martingale such that $M_0 = 1$ and $M_t = 0$ for all $t \geq 1$, and that M^t is a martingale for each $t \in [0, 1)$. Consider the process $U_t = M_t + t$ on $[0, 1]$ (note that U is not bounded in this case). Show that $V_t = M_t + 1$ on $[0, 1)$ and $V_1 = U_1 = 1$. Deduce that $\tau = 1$ is not optimal.

Exercise 13.2 Consider a Bachelier model with riskless asset of constant price 1 and a one-dimensional risky asset of price

$$S_t = S_0 + \sigma B_t,$$

for some constant S_0.

Consider the payoff process

$$U_t = g\left(S_t, Y_t\right),$$

for g a non-negative bounded measurable function of the asset price S and its modified running average $Y_t = \frac{1}{t+1}\left(Y_0 + \int_0^t S_u du\right)$ (for some constant Y_0).

(a) Argue why the value of the American option associated with U can be expressed as

$$\pi_t^A(U) = f\left(t, S_t, Y_t\right)$$

for some function f.

(b) Assuming that f is smooth enough, write a free boundary partial differential equation for f.

(c) Suppose that g is smooth. Find a condition that characterises an optimal exercise time $\tau < T$ for U, in terms of the derivatives of g.

(d) Let $g(S_t, Y_t) = (S_t - Y_t)^2$. Compute (heuristically) the optimal exercise time τ.

Exercise 13.3 Consider a market model where the n-dimensional stock price process X satisfies

$$d \log X_i(t) = \gamma_i(t)dt + \sum_{\nu=1}^{n} \sigma_{i\nu} dW_\nu(t),$$

where the γ_i and $\sigma_{i\nu}$ are progressively measurable processes satisfying appropriate integrability conditions.

(a) Define the market portfolio μ and what it means for a portfolio π to be functionally generated by a function S.

(b) Write down the formula for the portfolio π generated by S, a positive C^2 function defined on a neighbourhood U of the simplex Δ^n such that for each i, $x_i D_i \log S(x)$ is bounded on Δ^n.

(c) Compute the portfolios generated by the following functions:

- $S(x) = 1$.
- $S(x) = w_1 x_1 + \ldots + w_n x_n$, where the w_i are non-negative and not all equal to 0.
- $S(x) = x_1^{p_1} \ldots x_n^{p_n}$, where the p_i are constants adding up to 1.
- $S(x) = (w_1 x_1^p + \ldots + w_n x_n^p)^{1/p}$, where the w_i are non-negative and not all equal to 0 and $p > 0$.