Mathematical Finance

Exercise sheet 2

Exercise 2.1 Let $\mathbb{S}$ denote the family of simple predictable processes H, i.e.

$$H = H_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} H_i \mathbb{1}_{(\tau_i, \tau_{i+1}]}$$

for stopping times $0 = \tau_0 < \tau_1 < ... < \tau_{i+1} < \infty$ and bounded $\mathcal{F}_{\tau_i}$-measurable H_i for $i = 0, 1, ..., n+1$. Let $\mathbb{D}$ denote the family of adapted càdlàg processes and $\mathbb{L}$ denote the family of adapted càglàd processes on $[0, \infty)$. We endow $\mathbb{D}$ and $\mathbb{L}$ with the topology of convergence uniformly on compacts in probability, generated by the metric

$$d(X, Y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \mathbb{E}[|(X - Y)|_k^* \wedge 1].$$

Moreover, let the space of all measurable random variables L^0 be endowed with the topology generated by convergence in probability. Show that:

(a) The vector spaces $\mathbb{L}$ and $\mathbb{D}$ are complete.

(b) For some càdlàg process X, the following are equivalent:

1. The map $J_X : \mathbb{S} \to \mathbb{D}$ with $J_X(H) := H_0 X_0 + \sum_{i=1}^{n} H_i (X_{\tau_{i+1} \wedge \cdot} - X_{\tau_i \wedge \cdot})$, for $H \in \mathbb{S}$, is continuous with respect to the u.c.p. metric on $\mathbb{S}$ and $\mathbb{D}$, in other words, X is a good integrator.
2. For every $t \in [0, \infty)$, the mapping $I_{X^t} : \mathbb{S} \to L^0$ with $I_{X^t}(H) := J_X(H)_t$, for $H \in \mathbb{S}$, is continuous with respect to the uniform norm metric on $\mathbb{S}$.

Exercise 2.2 Prove that the set of good integrators is a vector space and show that it is generically not closed with respect to the ucp topology. Construct in particular examples of processes which are not good integrators but can be approximated by good integrators.

Exercise 2.3

(a) Let x be a càdlàg function on $[0, 1]$, and let π^n be a refining sequence of dyadic rational partitions of $[0, 1]$ with $\lim_{n\to\infty} \text{mesh}(\pi^n) = 0$. Show that, if the sum

$$\sum_{t_k^n, t_{k+1}^n \in \pi^n} y(t_k^n)(x(t_{k+1}^n) - x(t_k^n))$$

converges to a finite limit for every càglàd function y on $[0, 1]$, then x is of finite variation.

(b) Let X be a good integrator, and let Π^n be a sequence of partitions tending to identity. Show that

$$\sum_{\tau_k^n, \tau_{k+1}^n \in \Pi^n} Y_{\tau_k^n} (X_{\tau_{k+1}^n} - X_{\tau_k^n}) \overset{\text{ucp}}{\to} (Y \bullet X)$$

for every adapted càglàd process Y.