Mathematical Finance

Exercise sheet 3

Exercise 3.1 Let W^1, W^2 be two independent Brownian motions.

(a) For two C^2 functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, let $X_t = f(W_t^1, W_t^2)$ and $Y_t = g(W_t^1, W_t^2)$. Compute the quadratic covariation of X and Y.

(b) Show that $e^{W_t^1} \cos(W_t^2)$ is a martingale.

(c) Let $Z_t = \int_0^t \left(\cos(s) dW_s^1 + \sin(s) dW_s^2\right)$. Prove that Z is a Brownian motion.

Exercise 3.2 Let $N^1, ..., N^m$ and $W^1, ..., W^m$ all be independent, with each N^k a Poisson process of rate 1 and each W^k a Brownian motion, all starting at 0. Let $X^k = N^k + W^k$ for each k.

(a) Recall the formula for the stochastic exponential of a process given in the lecture notes. Find the stochastic exponential Z of $X = \sum_{k=1}^m X^k$, and check directly that it satisfies the stochastic differential equation (SDE)

$$dZ = Z_- dX, \quad Z_0 = 1.$$

By this we mean that the integrated form of this equation holds:

$$Z_t - 1 = (Z_- \bullet X)_t.$$

(b) Use Itô's formula to find a decomposition for the process

$$Y_t = |\mathbf{X}_t|^{2\alpha},$$

where $\mathbf{X}_t = (X_t^1, ..., X_t^m)$ and $|\mathbf{X}_t| = \left(\sum_{k=1}^m (X_t^k)^2\right)^{\frac{1}{2}}$, and we also assume $\alpha \in \mathbb{N}$.

(c) (optional) Let $\mathbf{v} \in \mathbb{R}^m$ and suppose that $P(\forall t \geq 0 \ \ \mathbf{X}_t, \mathbf{X}_{t-} \neq \mathbf{v}) = 1$.

Find a similar decomposition for the process

$$\tilde{Y}_t = |\mathbf{X}_t - \mathbf{v}|^{2\alpha},$$

where now $\alpha \in \mathbb{R}$.

Exercise 3.3 Let X be a continuous semimartingale, and let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Show that there exists a continuous increasing process A^f, with $A_0^f = 0$, such that

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s) dX_s + A_t^f$$

for all $t \in \mathbb{R}_+$ almost surely, where f'_- is the left derivative of f.