Mathematical Finance

Exercise sheet 4

Exercise 4.1

(a) Let W be a Brownian motion and τ an independent random variable taking non-negative real values. Consider the process
$$X = \mathcal{E}(W)^{\tau}.$$
Show that there exists a suitable choice of τ such that X is a uniformly integrable martingale but X^*_∞ is not integrable.

(b) Let $T \in (0, \infty)$ be the time horizon, L^∞ denote the class of all bounded martingales and H^∞ the class of martingales M such that $[M]_T$ is bounded. Show that $L^\infty \not\subseteq H^\infty$ and $H^\infty \not\subseteq L^\infty$.

(c) For a martingale M on $[0, T]$, denote
$$\|M\|_{BMO_2} := \sup_t \|E[|M_T - M_{t-}|^2 \mid \mathcal{F}_t]^{1/2}\|_\infty.$$
Let BMO be the set of martingales such that $\|M\|_{BMO_2} < \infty$. Show that $L^\infty, H^\infty \subseteq BMO$.

(d) Let H^1 denote the class of martingales with integrable maximum. Show that for $M \in H^1$ and $N \in BMO$, and assuming that M and N are continuous,
$$E\left[\int_0^T |d\langle M, N\rangle_s|\right] \leq c\|M\|_{H_1}\|N\|_{BMO_2}.$$

Exercise 4.2 Let B be a Brownian motion on $\mathbb{R}$ (starting at 0). For $x \in [-1, 1]$, we consider $B^x_t = x + B_t$, a Brownian motion "started at x". Let $\tau^x := \inf\{t > 0 : |B^x_t| \geq 1\}$ be the first time that it exits $[-1, 1]$.

(a) Let g be a continuous function on $[-1, 1]$. Show that the function $u : [-1, 1] \to \mathbb{R}$ defined by
$$u(x) = E\left[\int_0^{\tau_x} g(B^x_s) ds\right]$$
is well-defined and continuous.

Hint: Start by showing that τ_x is integrable by considering the martingale $(B^x)^2_t - t$.

(b) Suppose that v is a bounded function on $[-1, 1]$ such that $v(-1) = v(1) = 0$, and furthermore the process M^x defined by
$$M^x_t = v(B^x_{t\wedge\tau^x}) + \int_0^{t\wedge\tau^x} g(B^x_s) ds$$
is a local martingale for each x.

Prove that $u = v$.

(c) Suppose that v is a bounded function on $[-1, 1]$ such that $v(-1) = v(1) = 0$ and it satisfies the second-order differential equation
$$\frac{1}{2}v''(x) = -g(x). \tag{1}$$
Show that $v = u$.

(d) Replacing g by the Dirac delta mass δ_y at some point $y \in \mathbb{R}$, formally compute the solution v_y to (1). The function $v_y(x) =: G(x, y)$ is called the Green's function. Can you find a solution to (1) for more general g, in terms of G?

Exercise 4.3

(a) Let σ be a continuous positive function on $\mathbb{R}$, satisfying the linear growth condition:

$$|\sigma(x)| \leq K(1 + |x|)$$

for some $K > 0$. Suppose that we have a Brownian motion B and a family of processes X^x (for $x \in \mathbb{R}$) such that, for each $x \in \mathbb{R}$, the following stochastic differential equation is satisfied for all $t \geq 0$:

$$X_t^x = x + \int_0^t \sigma(X_s^x) dB_s.$$

Prove that for each time $T > 0$ there is a constant c (depending only on T, K and p but not on x) such that

$$E[((X_T^x)^*)^p] \leq c(1 + |x|^p).$$

(b) Construct a pair (X, B), where B is a Brownian motion, such that the following stochastic differential equation is satisfied:

$$X_t = \int_0^t \operatorname{sgn}(X_s) dB_s,$$

where $\operatorname{sgn}(x) = -\mathbb{1}_{x \leq 0} + \mathbb{1}_{x > 0}$.

Exercise 4.4 (Python) Simulate a random walk $(M_n)_{n \in \mathbb{N}}$ up to time 1000, starting from 0 and with the same probability $\frac{1}{2}$ of jumping up or down (by 1) at each step.

Quoting from [1], give explicit predictable integrands g and h and constants $c_p, C_p > 0$ such that the inequalities

$$(h \bullet M)_n + c_p [M, M]_n^{\frac{3}{2}} \leq (|M|_n^*)^3 \leq C_p [M, M]_n^{\frac{3}{2}} + (g \bullet M)_n$$

hold.

Compute the values taken by these processes along your simulated random walk, and plot them together with the process M_n^3.

References

[1] Beiglböck, Mathias; Siorpaes, Pietro. *Pathwise versions of the Burkholder–Davis–Gundy inequality.* Bernoulli 21 (2015), no. 1, 360–373. doi:10.3150/13-BEJ570. https://projecteuclid.org/euclid.bj/1426597073