Mathematical Finance

Exercise sheet 5

Exercise 5.1

(a) Let ϕ be a continuous process and M a continuous local martingale started at 0. Prove that

$$(\phi \bullet M)_{\cdot} = \int_0^{\cdot} \phi_s dM_s$$

is a local martingale started at 0.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic base, such that there are two independent stopping times T and U defined on it, which are independent and have an exponential distribution Exp(1). Define M by

$$M_t = \begin{cases} 0, & t < T \wedge U, \\ 1, & t \geq T \wedge U = T, \\ -1, & t \geq T \wedge U = U. \end{cases}$$

In other words, M starts at 0 and jumps once one of the stopping times arrive, with the jump being either to 1 or -1 depending on which stopping time arrives first. In the (probability 0) event that the two arrive simultaneously, M can be defined arbitrarily, i.e. we can take it to stay at 0.

Prove that M is a martingale with respect to its natural filtration $\mathcal{F}^M$. Prove that, for $\phi_t = \frac{1}{t}\mathbb{1}_{t>0}$ (which is predictable), $\phi \cdot M$ is not a local martingale with respect to $\mathcal{F}^M$.

Let $\mathcal{S}$ denote the set of semimartingales and $\mathbb{S}_1 := \{H \in \mathbb{S} : ||H||_\infty \leq 1\}$ the unit ball of simple predictable processes. The Emery topology is a topology on $\mathcal{S}$ generated by the metric

$$d_E(X,Y) := \sum_{n=1}^{\infty} 2^{-n} \sup_{H \in \mathbb{S}_1} E\left[1 \wedge \sup_{t \leq n} |(H \bullet (X-Y))_t|\right].$$

Exercise 5.2 Show that

(a) $\mathcal{S}$ endowed with the Emery topology is a topological vector space.

(b) $\mathcal{S}$ is closed in the Emery topology and complete with respect to the metric d_E.

(c) Show that the Emery topology is invariant under an equivalent change of measure.

(d) Let the set of adapted càglàd processes $\mathbb{L}$ be endowed with the u.c.p. topology and the set of semimartingales $\mathcal{S}$ be endowed with the Emery topology, and let X be a semimartingale. Show that

$$J_X : \mathbb{L} \ni Y \mapsto (Y \bullet X) \in \mathcal{S}$$

is continuous.

Exercise 5.3 Define fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ as a Gaussian process $(X_t)_{t \in \mathbb{R}_+}$ such $E[X_t] = 0$ for all $t \geq 0$ and the covariance function is given by

$$E[X_t X_s] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

for all $t, s \geq 0$.

We take a continuous version of X and denote it by W^H.

(a) Check that:

- The formula for the covariance is equivalent to the condition

$$E[|X_t - X_s|^2] = |t-s|^{2H}$$

for $t, s \geq 0$, together with $X_0 = 0$ almost surely.
- For $c > 0$, $(\frac{1}{c^H} W^H_{ct})_{t \geq 0}$ is a fBm of Hurst parameter H.
- For $t_0 > 0$, $(W^H_{t+t_0} - W^H_{t_0})_{t \geq 0}$ is a fBm of Hurst parameter H.
- For $H = \frac{1}{2}$, W^H is a Brownian motion.

(b) Use Birkhoff's ergodic theorem to compute the almost sure limit

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} |W^H_{k+1} - W^H_k|^p$$

for $p > 0$.

(c) Deduce that, for $H < \frac{1}{2}$, W^H has infinite quadratic variation.

Exercise 5.4 Consider a probability space $(\Omega, \mathcal{F}, P)$, together with a d-dimensional Brownian motion $(B_t)_{t\in[0,T]}$. Consider the natural filtration $\mathcal{F}^B_t = \mathcal{F}_t$ generated by B, and suppose that $\mathcal{F}_T = \mathcal{F}$.

(a) Show that any absolutely continuous measure $Q \ll P$ has a Radon-Nikodym derivative of the form

$$\frac{dQ}{dP} = \exp\left(\int_0^T \lambda_s dB_s - \frac{1}{2}\int_0^T ||\lambda_s||^2 ds\right)$$

for some $\lambda \in L(B)$.

Hint: You may use the Itô martingale representation theorem.

(b) For Q given in the above form, and assuming that $Q \sim P$, find (with proof) a d-dimensional Brownian motion under Q.

Hint: You may use the Girsanov-Meyer theorem.