Mathematical Finance

Exercise sheet 6

Exercise 6.1 We say that S is a special semimartingale if it can be decomposed as

$$S_t = S_0 + M_t + A_t,$$

where M is a local martingale and A is a predictable process of finite variation.

(a) Show that if S is a special semimartingale then the process $Y_t = \sup_{u\in[0,t]} |S_u - S_0|$ is locally integrable.

(b) Give an example of a semimartingale S which is not special.

Exercise 6.2 Let $W = W_0 + (W^1, W^2, W^2)$ be a Brownian motion in $\mathbb{R}^3$, i.e. W^1, W^2, W^3 are independent Brownian motions and $W_0 \in \mathbb{R}^3 \setminus \{0\}$ is an $\mathcal{F}_0$-measurable random variable.

(a) Show that $Y_t = |W_t|^{-1}$ is a local martingale as well as a supermartingale, where $|(x, y, z)| = \sqrt{x^2 + y^2 + z^2}$ is the Euclidean norm. You may assume that $P(\forall t\ W_t \neq 0) = 1$.

(b) Assume that W_0 is a standard normal random variable on $\mathbb{R}^3$. Show by direct calculation that $E[Y_t^2] = \frac{1}{t+1}$ for $t > 0$.

(c) Using the martingale convergence theorem, conclude that Y is not a martingale.

Exercise 6.3 Let X_k be independent Bernoulli random variables with $P(X_k = +1) = P(X_k = -1) = \frac{1}{2}$, $k \in \mathbb{N}$. Consider an infinite horizon model with a constant bank account normalized to one and a stock $S = (S_k)_{k\in\mathbb{N}}$ whose price is given by $S_0 = 1$, $S_k = S_{k-1} + X_k$, $k \in \mathbb{N}$. Consider the following strategy. Start with zero initial wealth, buy one stock and keep doubling your stock holdings until the stock goes up for the first time, then sell the stocks.

(a) Find the self-financing strategy $\varphi = (\eta, \vartheta)$ and the associated wealth process $V = (V_k(\varphi))_{k\in\mathbb{N}}$ with zero initial wealth for this strategy.

(b) Show that with this strategy, $V_\infty(\varphi) := \lim_{k\to\infty} V_k(\varphi) = 1$ a.s..

(c) Put

$$Y_t := \begin{cases} V_k(\varphi) & \text{for } 1 - \frac{1}{k+1} \le t < 1 - \frac{1}{k+2}, \\ 1 & \text{for } t \ge 1. \end{cases}$$

Assume the (augmented) natural filtration and show that the process Y is a local martingale, but not a martingale.

Exercise 6.4 Consider a financial market on the time interval $[0, T]$ consisting of one numéraire process with constant value $B_t = 1$ and two risky assets with dynamics given by geometric Brownian motion, i.e.

$$\begin{aligned} dS_t^1 &= S_t^1(\mu_1 dt + \sigma_1 dW_t^1), \quad S_0^1 = 1 \\ dS_t^2 &= S_t^2(\mu_2 dt + \sigma_2 dW_t^2), \quad S_0^2 = 1, \end{aligned}$$

where $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 > 0$ and W^1, W^2 are two Brownian motions with $[W^1, W^2]_t = \rho t$, for any $t \in [0, T]$ and a fixed $\rho \in [-1, 1]$.

(a) Find a condition on the coefficients μ_i, σ_i, ρ that is equivalent to this financial market being arbitrage-free.

(b) Under such conditions, and assuming that the filtration is the one generated by W^1 and W^2, find all the equivalent martingale measures for this financial market.

(c) (optional) Can you repeat this exercise if we assume that the correlation ρ is random and time-dependent, i.e. $[W^1, W^2]_t = \int_0^t \rho_s ds$ for some continuous process ρ_s taking values in $[-1, 1]$?