Mathematical Finance

Exercise sheet 8

Exercise 8.1 As in exercise 2 of sheet 6, let $W = W_0 + (W^1, W^2, W^3)$ be a Brownian motion in $\mathbb{R}^3$, i.e. W^1, W^2, W^3 are independent Brownian motions and $W_0 \in \mathbb{R}^3 \setminus \{0\}$ is an $\mathcal{F}_0$-measurable random variable.

Consider the market defined by a riskless asset S^0 with constant price 1, and the risky asset with price process

$$S_t = |W_t|.$$

Show that this market satisfies (NUPBR) but not (NA).

Exercise 8.2 Suppose we define a model with time interval $[0,1]$, one riskless asset of constant price 1 and one risky asset which is a compound Poisson process with standard normal jumps.

Specifically, for some Poisson process $(N_t)_{t\in[0,1]}$ of rate 1 and $(Z_k)_{k\in\mathbb{N}}$ a sequence of i.i.d. standard normal variables (also independent from N), we have that

$$S_t = \sum_{k=1}^{N_t} Z_k.$$

We take the natural filtration of S.

Show that the only admissible strategy is 0.

Exercise 8.3 Consider a discrete time setting with deterministic time points $0 = t_0 < t_1 < t_2 < ... < t_n = T$. In this setting, semimartingales are given in the form

$$S = \sum_{k=0}^{n-1} S_k \mathbb{1}_{[t_k, t_{k+1})} + S_n \mathbb{1}_{\{T\}},$$

where each S_k is $\mathcal{F}_{t_k}$-measurable.

Show that in this case, ucp convergence is equivalent to convergence in Emery topology.

Exercise 8.4 Show that

(a) A local martingale is a sigma-martingale.

(b) A sigma-martingale which is also a special semimartingale is a local martingale.

Exercise 8.5 In the same setup of question 1, consider the Bachelier model:

$$S_t = S_0 + \mu t + \sigma B_t$$

on $[0,T]$, where B is a d-dimensional Brownian motion, $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d\times d}$ is invertible.

(a) Show that there exists a unique equivalent martingale measure Q such that for all $f \in L^\infty(\mathcal{F}_T)$, $E_Q(f) = \pi(f)$, where π is the superreplication price.

(b) Take $d = 1$ and $f = (S_T - K)^+$, for some $K \in \mathbb{R}$. Compute $\pi(f)$ as well as the unique strategy ϑ such that

$$\pi(f) + (\vartheta \bullet S)_T = f.$$

(c) Have a look at this paper and write a very short summary of some of the main points.

Exercise 8.6 (Python) Assume Black-Scholes dynamics for S, say $(r, \mu, \sigma) = (0, 0, 1)$, and find the hedging strategy H for the log-contract g whose discounted payoff is given by

$$g(S_T) = \log \frac{S_T}{S_0} + \frac{1}{2}\sigma^2 T.$$

Compare numerically the value of $g(S_T)$ to $(H \bullet S)_T$ at $T = 1$.

References

[1] Walter Schachermayer; Josef Teichmann. *How close are the option pricing formulas of Bachelier and Black-Merton-Scholes?* Mathematical Finance, 18: 155-170. doi:10.1111/j.1467-9965.2007.00326.x