Mathematical Finance

Exercise sheet 9

Exercise 9.1

(a) Let $\mathcal{C} \subseteq L^0$ be non-empty, closed, convex and bounded. Suppose that $J : L^0 \to \mathbb{R}$ is a continuous strictly concave function such that

$$\sup_{g \in \mathcal{C}} J(g) < \infty,$$

and that $J \not\equiv -\infty$ on $\mathcal{C}$. Show that J has a unique maximiser $\hat{g}$.

(b) Let $(\mathcal{C}_n)_{n\in\mathbb{Z}}$ be an increasing sequence of closed, convex, bounded subsets of L^0, i.e. such that $\mathcal{C}_n \subseteq \mathcal{C}_m$ for $n \leq m$. Show that J has a unique maximiser $\hat{g}_{-\infty}$ on

$$\mathcal{C}_{-\infty} := \bigcap_{n\in\mathbb{Z}} \mathcal{C}_n$$

if $\mathcal{C}_{-\infty}$ is non-empty and $J \not\equiv -\infty$ on $\mathcal{C}_{-\infty}$, and moreover there exists a sequence $(\tilde{h}_n)$ of forward convex combinations of $(\hat{g}_{-n})_{n\geq 1}$ such that $\tilde{h}_n \to \hat{g}_{-\infty}$ as $n \to \infty$.

(c) Suppose that $\bigcup_{n\in\mathbb{Z}} \mathcal{C}_n$ is bounded in probability, and that f is bounded above in this set. Show that J has a unique maximiser $\hat{g}_\infty$ on

$$\mathcal{C}_\infty = \overline{\bigcup_{n\in\mathbb{Z}} \mathcal{C}_n}^{L^0},$$

and moreover $\hat{g}_n \to \hat{g}_\infty$ as $n \to \infty$.

(d) Suppose that J is uniformly strictly concave, in the sense that there exists a continuous strictly increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(0) = 0$ and such that for $f_1, f_2 \in L^0$,

$$J\left(\frac{f_1 + f_2}{2}\right) - \frac{J(f_1) + J(f_2)}{2} \geq \rho(d(f_1, f_2)).$$

Show that in (b) and (c), we already have that $\hat{g}_{-n} \to g_{-\infty}$ and $\hat{g}_n \to g_\infty$, respectively.

Exercise 9.2

(a) Consider the Bachelier model

$$S_t = S_0 + \sigma B_t$$

in its natural filtration on the interval $[0, T]$. Prove the martingale representation theorem for bounded martingales, using the fact that the set of equivalent separating measures is a singleton.

(b) Consider a continuous trajectory model S for a d-dimensional discounted price process in its natural filtration. Assume that there is only one equivalent separating measure. Prove that martingale representation holds true for bounded martingales.

(c) Consider finite filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supporting a model S for a d-dimensional discounted price process. Assume that there exists more than one equivalent separating measure. Prove that there is at least one bounded claim g such that either g or $-g$ cannot be replicated.

Exercise 9.3 Consider a general model, with $[0,1]$ as the time interval, a riskless asset of constant price 1, and some d-dimensional semimartingale S representing the prices of the risky assets.

Define

$$G = \{(\vartheta \bullet S)_T, \vartheta \in \Theta_{adm}\} \subseteq L^0$$

and

$$C = (G - L^0_{\geq 0}) \cap L^\infty \subseteq L^\infty.$$

(a) Show that the following notions of no arbitrage are equivalent:

$$G \cap L^0_{\geq 0} = \{0\}$$

and

$$C \cap L^\infty_{\geq 0} = \{0\}.$$

(b) Prove that C is weak-$*$-closed if and only for any bounded sequence (f_n) in C converging almost surely to f_0, it holds that $f_0 \in C$.