Mathematical Finance

Exercise sheet 1

Exercise 1.1 Let $(M_n)_{n\in\mathbb{N}}$ be a martingale such that $M_0 = 0$ and

$$|M_n - M_{n-1}| \le a_n \quad P\text{-a.s.}$$

for each n and a sequence (a_n) of non-negative constants, with $\sum_{i=1}^{\infty} a_i^2 = A^2 < \infty$.

(a) Prove that M is bounded in L^2. Deduce that $M_n \to M_\infty$ almost surely and in L^2, for some M_∞ in L^2.

(b) Show that

$$P\left(\sup_{k\ge 0} M_k \ge c\right) \le \exp\left(-\frac{c^2}{2A^2}\right),$$

for any $c > 0$.

Hint: Try applying Doob's maximal inequality to $(e^{\lambda M_n})$, for some $\lambda > 0$. You may use the inequality $\cosh(x) \le e^{x^2/2}$ (for $x \in \mathbb{R}$).

Solution 1.1

(a) Recall the simple fact that, since M is a martingale,

$$E[(M_{n+1} - M_n)^2 \mid \mathcal{F}_n] = E[M_{n+1}^2 - M_n^2 \mid \mathcal{F}_n].$$

From this and $M_0 = 0$ it follows that

$$\begin{aligned} E[M_n^2] &= \sum_{i=1}^{n} E[(M_n - M_{n-1})^2] \\ &\le \sum_{i=1}^{n} a_i^2 \le A^2 < \infty \end{aligned}$$

using the assumptions. Therefore M is bounded in L^2. It is known that boundedness in L^2 implies in particular uniform integrability, so by the martingale convergence theorem there is a limit $M_n \to M_\infty$ almost surely and in L^1. Boundedness of M in L^2 implies furthermore that M_∞ is in L^2 and that the convergence also happens in L^2.

(b) Let $\lambda > 0$ be fixed and let

$$Z_n = e^{\lambda M_n}.$$

Then we have the following (note that, since Z is non-negative, we don't need to assume integrability):

$$E[Z_n \mid \mathcal{F}_{n-1}] = Z_{n-1} E[e^{\lambda(M_n - M_{n-1})} \mid \mathcal{F}_{n-1}].$$

To estimate this term, note that $|M_n - M_{n-1}| \le a_n$ by assumption. On $[-a_n, a_n]$ we have (by convexity) the inequality

$$\frac{a_n - x}{2a_n} e^{-\lambda a_n} + \frac{a_n + x}{2a_n} e^{\lambda a_n} \geq e^{\lambda x}$$

(this simply follows from convexity). Thus,

$$\begin{aligned}
E[Z_n \mid \mathcal{F}_{n-1}] &\leq Z_{n-1} \left(\frac{a_n - E[M_n - M_{n-1} \mid \mathcal{F}_{n-1}]}{2a_n} e^{-\lambda a_n} + \frac{a_n + E[M_n - M_{n-1} \mid \mathcal{F}_{n-1}]}{2a_n} e^{\lambda a_n} \right) \\
&= Z_{n-1} \left(\frac{1}{2} e^{-\lambda a_n} + \frac{1}{2} e^{\lambda a_n} \right) \\
&= Z_{n-1} \cosh(\lambda a_n) \\
&\leq Z_{n-1} e^{\lambda^2 a_n^2 / 2},
\end{aligned}$$

using that M is a martingale and the given inequality.

Iterating, we obtain

$$E[Z_n] \leq \exp\left(\lambda^2 \sum_{i=1}^{n} a_i^2 / 2 \right) \leq \exp(\lambda^2 A^2 / 2).$$

In particular, this proves that Z_n is integrable, and from Jensen's inequality it follows easily that Z is a submartingale (since M is a martingale).

Next, we apply Doob's maximal inequality to Z to obtain the following (let $M_n^* = \max_{0 \leq k \leq n} M_k$, etc):

$$\begin{aligned}
P(M_n^* \geq c) &= P(Z_n^* \geq e^{\lambda c}) \\
&\leq e^{-\lambda c} E[Z_n] \\
&\leq e^{-\lambda c + \lambda^2 A^2 / 2}.
\end{aligned}$$

At this point we haven't specified what value $\lambda > 0$ will take, and so we are free to choose a convenient one. We choose λ so as to minimise the exponent, meaning that $\lambda = \frac{c}{A^2}$ and so

$$P(M_n^* \geq c) \leq \exp\left(-\frac{c^2}{2A^2} \right),$$

which is precisely the bound we want. To replace M_n^* with M_∞^* we simply use the monotone convergence theorem.

Exercise 1.2 Let μ be a probability measure on $(0, +\infty)$. Consider (on some probability space) independent $N, Y_1, Y_2, Y_3, ...$ where each Y_i has distribution μ and $N = (N_t)_{t\in[0,1]}$ is a Poisson process on $[0, 1]$ of rate $\lambda > 0$. Consider the compound Poisson process X on $[0, 1]$ given by

$$X_t = \sum_{i=1}^{N_t} Y_i.$$

(a) Find a necessary and sufficient condition for X to be a submartingale with respect to its natural filtration.

(b) Show that under that condition, X is a submartingale of class (D). Find a decomposition

$$X_t = M_t + A_t \quad \forall t \in [0, 1],$$

where M is a martingale and A is an increasing predictable process, both with càdlàg trajectories.

(c) Show through direct calculations that X is a good integrator.

Solution 1.2

(a) Since the Y_i are non-negative, the following computations hold for $t \in [0, 1]$:

$$\begin{aligned}
E[X_t] &= E\left[\sum_{i=1}^{N_t} Y_i\right] \\
&= E\left[E\left[\sum_{i=1}^{N_t} Y_i \mid N_t\right]\right] \\
&= E\left[\sum_{i=1}^{N_t} E\left[Y_i \mid N_t\right]\right] \\
&= E\left[N_t \int_{(0,\infty)} x d\mu(x)\right] \\
&= \lambda t \int_{(0,\infty)} x d\mu(x),
\end{aligned}$$

using independence and the distribution of the Y_i.

Therefore, if X is going to be a submartingale then $\int_{(0,\infty)} x d\mu(x) =: \mu_0 < \infty$ is required. This is in fact sufficient: the calculations above show that X is integrable. It is obviously adapted to its natural filtration, and since it is (almost surely) increasing it must be a submartingale.

(b) Note that since X is increasing, for any stopping time τ (taking values on $[0, 1]$), $X_\tau \le X_1$. Since X_1 is integrable (and X_τ is non-negative), this means that $\{X_\tau : \tau \text{ is a stopping time on } [0, 1]\}$ is uniformly integrable. Therefore $(X_t)_{t\in[0,1]}$ is of class (D).

We find the required decomposition:

$$X_t = (X_t - \lambda\mu_0 t) + \lambda\mu_0 t.$$

Clearly this is a valid decomposition, and $A_t = \lambda\mu_0 t$ is increasing, predictable (even deterministic) and càdlàg. $M_t = X_t - \lambda\mu_0 t$ is clearly càdlàg since X is, and we want to show that it is a martingale.

Since X is adapted and integrable, so is M. To show that it is a martingale, note that

$$\begin{aligned}
E[M_t \mid F_s] &= M_s - \lambda(t-s) + E\left[\sum_{i=N_s+1}^{N_t} Y_i \mid \mathcal{F}_s\right] \\
&= M_s - \lambda\mu_0(t-s) + E\left[E\left[\sum_{i=N_s+1}^{N_t} Y_i \mid \sigma(N_t,\mathcal{F}_s)\right] \mid \mathcal{F}_s\right] \\
&= M_s - \lambda\mu_0(t-s) + E\left[\sum_{i=N_s+1}^{N_t} E\left[Y_i \mid \sigma(N_t,\mathcal{F}_s)\right] \mid \mathcal{F}_s\right] \\
&= M_s - \lambda\mu_0(t-s) + E\left[\sum_{i=N_s+1}^{N_t} \mu_0 \mid \mathcal{F}_s\right] \\
&= M_s - \lambda\mu_0(t-s) + E\left[\mu_0(N_t - N_s) \mid \mathcal{F}_s\right] \\
&= M_s,
\end{aligned}$$

as we wanted (using again independence, the distribution of the Y_i as well as independence of increments of the Poisson process).

(c) Given a simple integrand $H = H_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^n H_i \mathbb{1}_{(\tau_i,\tau_{i+1}]}$, we have that

$$\begin{aligned}
|(H \bullet X)_t| &= \left|\sum_{i=1}^n H_i (X_{\tau_{i+1}\wedge t} - X_{\tau_i \wedge t})\right| \\
&\le \sum_{i=1}^n |H_i| |X_{\tau_{i+1}\wedge t} - X_{\tau_i \wedge t}| \\
&\le \sum_{i=1}^n |H_i| (X_{\tau_{i+1}} - X_{\tau_i}) \\
&\le X_1 \sup_{t\in[0,1]} |H_t|
\end{aligned}$$

Now, if we take simple integrands H^k converging to 0 in ucp topology, then we have that for $\epsilon > 0$,

$$\begin{aligned}
P(\sup_{t\in[0,1]} |(H^k \bullet X)_t| > \epsilon) &\le P(\sup_{t\in[0,1]} |H^k_t| X_1 > \epsilon) \\
&\le P\left(\sup_{t\in[0,1]} |H^k_t| > \frac{\epsilon}{M}\right) + P(X_1 > \epsilon M)
\end{aligned}$$

for any $M > 0$. By choosing M large enough we can make the second term small, and then by choosing k large enough we can make the first term small as well (using ucp convergence of H^k to 0). This shows convergence of $H^k \bullet X$ to 0 in the ucp topology.

Exercise 1.3 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis satisfying the usual conditions. Let M be a local martingale and X an adapted càdlàg process. Prove the following statements:

(a) M is a uniformly integrable martingale if and only if it is of class (D).

(b) Let X have terminal value X_∞. Then X is a uniformly integrable martingale if and only if for all stopping times τ, the variable X_τ is integrable and $E[X_\tau] = E[X_0]$.

(c) Let X be predictable and τ a stopping time. Then $X_\tau \mathbb{1}_{\tau<\infty}$ is $\mathcal{F}_{\tau-}$-measurable.

(d) Let τ be a predictable finite stopping time. Then $M_{\tau-} = E[M_\tau \mid \mathcal{F}_{\tau-}]$.

(e) Let M be predictable. Then M is continuous.

Hint: For (d), you may use the fact that τ is a predictable stopping time if and only if there exists an *announcing sequence* $(\tau_n)_{n=1}^\infty$ for τ, defined as an increasing sequence of stopping times such that $\tau_n \uparrow \tau$ and $\tau_n < \tau$ P-a.s. on $\tau > 0$.

Solution 1.3

(a) If M is a uniformly integrable martingale, by the martingale convergence theorem there exists a limit $M_\infty \in L^1(\mathcal{F})$ such that $M_t = E[M_\infty \mid \mathcal{F}_t]$ for all $t \geq 0$. Indeed, we obtain $M_\tau = E[M_\infty \mid \mathcal{F}_\tau]$ for all stopping times τ, by the optional stopping theorem. Then, the family $\{M_\tau = E[M_\infty \mid \mathcal{F}_\tau], \tau \text{ a finite stopping time}\}$ is uniformly integrable, as a family of conditional expectations of an integrable random variable.

Conversely, let M be of class (D). Find a localising sequence (τ_n) for M, then for any $s < t$ we have

$$E[M_{t\wedge\tau_n} \mid \mathcal{F}_s] = M_{s\wedge\tau_n}.$$

Note that $M_{t\wedge\tau_n} \to M_t$ and $M_{s\wedge\tau_n} \to M_s$ P-almost surely as $n \to \infty$. Since $(M_{t\wedge\tau_n})_{n=1}^\infty$ is uniformly integrable, the convergence is also in L^1 (Vitali convergence theorem). Thus, we obtain $E[M_t \mid \mathcal{F}_s] = M_s$. Since M is integrable and adapted, it is a uniformly integrable martingale.

(b) If X is a uniformly integrable martingale, then X_τ is integrable and $E[X_\tau] = E[X_0]$ for any stopping time τ, by the optional stopping theorem (this also holds when $\tau = \infty$).

Suppose now that X_τ is integrable and $E[X_\tau] = E[X_0]$ for all stopping times τ. By assumption, X_t is in particular integrable for each $t \in [0, \infty]$. To show that X is a uniformly integrable martingale, it suffices to show that $X_t = E[X_\infty \mid \mathcal{F}_t]$, for each $t \geq 0$.

By definition of conditional expectation, it is enough to show that $E[X_t \mathbb{1}_A] = E[X_\infty \mathbb{1}_A]$ for any event $A \in \mathcal{F}_t$. By taking $\tau = t\mathbb{1}_A + \infty \mathbb{1}_{A^c}$, the equality holds from the assumption.

(c) We use the monotone class theorem. Let

$$\mathcal{H} := \{X : \Omega \times [0, \infty) \to \mathbb{R} \mid \forall \tau \text{ a stopping time}, X_\tau \mathbb{1}_{\tau<\infty} \text{ is } \mathcal{F}_{\tau\_}\text{-measurable}\}.$$

It is obvious that $\mathcal{H}$ is a vector space, and likewise for any increasing sequence $X^n \uparrow X$ of non-negative $X^n \in \mathcal{H}$, we have that

$$X_\tau \mathbb{1}_{\tau<\infty} = \lim_{n\to\infty} X_\tau^n \mathbb{1}_{\tau<\infty}$$

is $\mathcal{F}_{\tau\_}$-measurable.

Finally, we show that $H \in \mathcal{H}$, for any simple predictable process H of the form

$$H = \mathbb{1}_{]s,t]} \mathbb{1}_A$$

or

$$H = \mathbb{1}_{\{0\}}\mathbb{1}_A$$

where $A \in \mathcal{F}_s$ and $A \in \mathcal{F}_0$, respectively. This is enough, since processes of this form generate the predictable σ-algebra $\mathcal{P}$, so that $\mathcal{H}$ contains all predictable processes. We consider the first case, as the second is similar. Note that

$$H_\tau = \mathbb{1}_{s<\tau\leq t}\mathbb{1}_A = \mathbb{1}_{\{s<\tau\}\cap A}\mathbb{1}_{\tau\leq t}.$$

By definition of $\mathcal{F}_{\tau-}$, we have that $\{s < \tau\} \cap A \in \mathcal{F}_{\tau-}$ for $A \in \mathcal{F}_s$. Since $\{\tau \leq t\} = \{\tau > t\}^c \in \mathcal{F}_{\tau-}$ we conclude that H_τ is $\mathcal{F}_{\tau-}$, as we wanted.

(d) Assume first that M is a uniformly integrable martingale. From the hint, we may take an announcing sequence (τ_n) for τ. Since M is a uniformly integrable martingale, it holds by the optional stopping theorem that

$$M_{\tau_n} = E[M_\tau \mid \mathcal{F}_{\tau_n}]$$

for each n. Note that one can see $(M_{\tau_n})_{n=1}^\infty$ as a discrete uniformly integrable martingale with respect to the filtration $\mathcal{G}_n := \mathcal{F}_{\tau_n}$. The martingale convergence theorem gives that

$$M_{\tau_n} \to \eta$$

in L^1 for some integrable random variable η. Moreover, we have that η is $\mathcal{G}_\infty$-measurable, where $\mathcal{G}_\infty = \sigma\left(\bigcup_{n=1}^\infty \mathcal{G}_n\right)$, and indeed $\eta = E[M_\tau \mid \mathcal{G}_\infty]$. So we just need to show that $\mathcal{G}_\infty = \mathcal{F}_{\tau-}$, which is straightforward to check. We obtain

$$E[M_\tau \mid \mathcal{F}_{\tau-}] = \eta = \lim_{n\to\infty} M_{\tau_n} = M_{\tau-}.$$

Now, we consider the general case. Let M be a local martingale and (T_n) a localising sequence. We have that

$$\begin{aligned}
E[M_\tau \mid \mathcal{F}_{\tau-}]\mathbb{1}_{\tau\leq T_n} &= E[M_\tau \mathbb{1}_{\tau\leq T_n} \mid \mathcal{F}_{\tau-}] \\
&= E[M_{\tau\wedge T_n}\mathbb{1}_{\tau\leq T_n} \mid \mathcal{F}_{\tau-}] \\
&= E[M_{\tau\wedge T_n} \mid \mathcal{F}_{\tau-}]\mathbb{1}_{\tau\leq T_n} \\
&= M_{\tau-\wedge T_n}\mathbb{1}_{\tau\leq T_n} \\
&= M_{\tau-}\mathbb{1}_{\tau\leq T_n},
\end{aligned}$$

using that $\{\tau \leq T_n\} \in \mathcal{F}_{\tau-}$ and that we proved the result for uniformly integrable martingales. Therefore, $E[M_\tau \mid \mathcal{F}_{\tau-}] = M_{\tau-}$ on $\{\tau \leq T_n\}$. Since $\bigcup_{n=1}^\infty \{\tau \leq T_n\} = \Omega$ up to a null set, we have that $E[M_\tau \mid \mathcal{F}_{\tau-}] = M_{\tau-}$ P-a.s..

(e) If M is a predictable local martingale, we can apply both results (c) and (d). Since M is predictable, for any finite stopping time we have that M_τ is $\mathcal{F}_{\tau-}$-measurable, so that $E[M_\tau \mid \mathcal{F}_{\tau-}] = M_\tau$ P-a.s. by (c). We also have that $M_{\tau-} = E[M_\tau \mid \mathcal{F}_{\tau-}]$ for any predictable finite stopping time by (d), so that $M_\tau = M_{\tau-}$ P-a.s. for any predictable finite stopping time τ. By the predictable section theorem it follows that M and M_- are indistinguishable, and so M is almost surely continuous (since we assume that it is càdlàg).

Exercise 1.4 Let A be an increasing locally integrable process with $A_0 = 0$. Show that, for an increasing predictable process A^p with $A^p_0 = 0$, the following conditions are equivalent:

1. $A - A^p$ is a local martingale;
2. $E[A^p_\tau] = E[A_\tau]$ for all stopping times τ;
3. $E[(H \bullet A^p)_\infty] = E[(H \bullet A)_\infty]$ for all nonnegative simple predictable processes H.

Show that there exists a unique such process A^p, known as the *dual predictable projection* or *compensator* of A.

Hint: Use the Doob-Meyer decomposition theorem to prove uniqueness and existence.

Solution 1.4 3. $\Rightarrow$ 2. is immediate by setting $H = \mathbb{1}_{[0,\tau[}$.

2. $\Rightarrow$ 1.: Let (τ_n) be a localising sequence for A, i.e. such that $E[A_{\tau_n}] = E[A^p_{\tau_n}] < \infty$. It follows from 3b) that $(A - A^p)^{\tau_n}$ is a uniformly integrable martingale, and so $A - A^p$ is a local martingale.

1. $\Rightarrow$ 3.: It is enough to check this for $H = H_s \mathbb{1}_{]s,t]}$ for $s < t$ and H_s a $\mathcal{F}_s$-measurable random variable. We need to check that

$$E[H_s(A^p_t - A^p_s)] = E[H_s(A_t - A_s)].$$

Let (τ_n) be a localising sequence for $A - A^p$, and we may assume that $A^{\tau_n}, (A^p)^{\tau_n}$ are integrable, i.e. $E[A_{\tau_n}], E[A^p_{\tau_n}] < \infty$. Since $(A - A^p)^{\tau_n}$ is a uniformly integrable martingale, we obtain that

$$E[H_s(A^p_{t\wedge\tau_n} - A^p_{s\wedge\tau_n})] = E[H_s(A_{t\wedge\tau_n} - A_{s\wedge\tau_n})]$$

(by taking a conditional expectation). Since $A_{t\wedge\tau_n} - A_{s\wedge\tau_n}$ and $A^p_{t\wedge\tau_n} - A^p_{s\wedge\tau_n}$ are increasing in n, and $H_s \geq 0$, we obtain the result by the monotone convergence theorem.

Finally, we show that a unique such A^p exists. Once again, let (τ_n) be a localising sequence such that $E[A_{\tau_n}] < \infty$. Since A is increasing, it is clear that A^{τ_n} is a submartingale of class (D) for each n, and therefore it has a unique Doob-Meyer decomposition

$$A^{\tau_n} = M^n + B^n,$$

where M^n is a local martingale and B^n is an increasing predictable process, both started at 0.

We note that the (M^n) and (B^n) are consistent, in the sense that $(M^n)^{\tau_{n-1}} = M^{n-1}$ and $(B^n)^{\tau_{n-1}} = B^{n-1}$ for each n. Indeed, we have that

$$A^{\tau_{n-1}} = M^{n-1} + B^{n-1}$$

and

$$A^{\tau_{n-1}} = (A^{\tau_n})^{\tau_{n-1}} = (M^n)^{\tau_{n-1}} + (B^n)^{\tau_{n-1}},$$

so that the uniqueness of the decomposition shows the consistency.

Next, we can define $M = M^n$ and $B = B^n$ on $[0, \tau_n[$: since $\tau_n \uparrow \infty$, this defines M and B up to a null set (the consistency of (M^n) and B^n) ensures that this definition make sense). We claim that $B = A^p$ is the process we want. It is clear from the definition that B is increasing and predictable since the B^n are, and $A - B = M$ is a local martingale since τ_n is a localising sequence. Therefore, 1. holds, as we wanted.

To see that B is the unique such process, let $\tilde{B}$ be another candidate. We can find a common localising sequence (τ_n) for $A - B$ and $A - \tilde{B}$. Then,

$$A^{\tau_n} = (A - B)^{\tau_n} + B^{\tau_n} = (A - \tilde{B})^{\tau_n} + \tilde{B}^{\tau_n}$$

are two Doob-Meyer decompositions for A^{τ_n}, so that $B^{\tau_n} = \tilde{B}^{\tau_n}$ for each n, which implies that $B = \tilde{B}$.