Mathematical Finance

Exercise sheet 11

Exercise 11.1 Consider a financial market S. Let

$$\mathcal{X}_1 = \{1 + \vartheta \bullet S : \vartheta \in \Theta^1_{\text{adm}}\}.$$

Show that $\mathcal{X}_1$ satisfies the switching property: for any $X \in \mathcal{X}_1$ and any strictly positive $X' \in \mathcal{X}_1$, a stopping time τ and event $A \in \mathcal{F}_\tau$, we have that

$$\tilde{X} = \mathbb{1}_{\Omega \setminus A} X_\cdot + \mathbb{1}_A \frac{X'_{\tau \vee \cdot}}{X'_\tau} X_{\tau \wedge \cdot}$$

belongs to $\mathcal{X}_1$.

Solution 11.1 Let $X, X' \in \mathcal{X}_1$, with $X' > 0$. Then, we can find $\vartheta, \vartheta' \in \Theta^1_{\text{adm}}$ such that $X = 1 + \vartheta \bullet S$ and $X' = 1 + \vartheta' \bullet S$.

Now, note that

$$\hat{X} := 1 + \left[\mathbb{1}_{[0,\tau]}\vartheta + (\mathbb{1}_{(\tau,T]}\mathbb{1}_{\Omega \setminus A}\vartheta) + \left(\mathbb{1}_{(\tau,T]}\mathbb{1}_A \frac{X_\tau}{X'_\tau}\vartheta'\right)\right] \bullet S = \tilde{X}.$$

Indeed, we can check that $\hat{X} = X = \tilde{X}$ on $\Omega \setminus A$ and on A if $0 \le t \le \tau$, while on A with $t \ge \tau$,

$$\begin{aligned}
\hat{X}_t &= 1 + \int_0^\tau \vartheta_s dS_s + \int_\tau^t \frac{X_\tau}{X'_\tau}\vartheta'_s dS_s \\
&= X_\tau \left(1 + \frac{1}{X'_\tau}\int_\tau^t \vartheta'_s dS_s\right) \\
&= X_\tau \left(1 + \frac{1}{X'_\tau}(X'_t - X'_\tau)\right) \\
&= \frac{X_\tau}{X'_\tau} X'_t = \tilde{X}_t.
\end{aligned}$$

Moreover, note that the strategy $\tilde{\vartheta} := \mathbb{1}_{[0,\tau]}\vartheta + (\mathbb{1}_{(\tau,T]}\mathbb{1}_{\Omega \setminus A}\vartheta) + \left(\mathbb{1}_{(\tau,T]}\mathbb{1}_A \frac{X_\tau}{X'_\tau}\vartheta'\right)$ is predictable, since each of ϑ, ϑ', $\mathbb{1}_{[0,\tau]}$, $\mathbb{1}_{(\tau,T]}\mathbb{1}_{\Omega \setminus A}$, $\mathbb{1}_{(\tau,T]}\mathbb{1}_A$, $\mathbb{1}_{(\tau,T]}X_\tau$, $\mathbb{1}_{(\tau,T]}\frac{1}{X'_\tau}$ is predictable. The fact that $\tilde{\theta} \in L(S)$ can be deduced easily from Theorem 1.29 in the lecture notes.

For the following questions, we consider a financial market satisfying (NFLVR), with a numéraire (which we set to be equal to 1) and a d-dimensional risky asset with discounted prices S_t taking values in $D \subseteq \mathbb{R}^d$.

We work under an ESM, Q, and we will work with polynomial models. Consider the following two models:

- The Black-Scholes model:

$$dS_t = S_t \sigma dW_t \quad (S_0 \in \mathbb{R}^d_{>0}),$$

  where $\sigma \in \mathbb{R}^{d\times d}$ is invertible, and W is a d-dimensional Brownian motion. S_t in the right-hand side is viewed as a diagonal matrix with entries S^i.

- The SABR model (in Bachelier form):

$$dS^1_t = S^2_t dW_t, \quad dS^2_t = \alpha S^2_t dB_t \quad (S^1_0, S^2_0 > 0),$$

  for some parameter $\alpha > 0$ and Brownian motions W, B with fixed correlation $\rho \in [-1, 1]$. S^2 is the stochastic volatility, which we assume that we can trade through forward volatility contracts.

For these two models, solve the following exercises:

Exercise 11.2 Show that S is a polynomial process, in other words, that for any $s \le t$ and any polynomial p of degree n we have

$$E[p(S_t) \mid \mathcal{F}_s] = q(S_s),$$

where q is a polynomial of degree $\le n$, whose coefficients are functions of $t - s$.

Solution 11.2 In the Black-Scholes, consider a monomial $x_1^{k_1} x_2^{k_2} ... x_d^{k_d}$. We compute

$$\begin{aligned}\prod_{j=1}^d (S^j_t)^{k_j} &= \prod_{j=1}^d (S^j_0)^{k_j} \exp\left(\left(\sum_{j=1}^d k_j \sigma_j\right) B_t - \frac{t}{2} \sum_{j=1}^d k_j |\sigma_j|^2\right) \\ &= \prod_{j=1}^d (S^j_s)^{k_j} \exp\left(\left(\sum_{j=1}^d k_j \sigma_j\right) (B_t - B_s) - \frac{t-s}{2} \sum_{j=1}^d k_j |\sigma_j|^2\right),\end{aligned}$$

where σ_j are the rows of σ, i.e. $\sigma_j = (\sigma_{j1}, ..., \sigma_{jd})$. This can be further decomposed as

$$\prod_{j=1}^d (S^j_t)^{k_j} = \prod_{j=1}^d (S^j_s)^{k_j} \exp\left(\left(\sum_{j=1}^d k_j \sigma_j\right) (B_t - B_s) - \frac{t-s}{2} \left|\sum_{j=1}^d k_j \sigma_j\right|^2\right) \exp\left(\frac{t-s}{2}\left(\left|\sum_{j=1}^d k_j \sigma_j\right|^2 - \sum_{j=1}^d k_j |\sigma_j|^2\right)\right).$$

Since the middle term is a martingale (by Novikov), we obtain

$$E\left[\prod_{j=1}^d (S^j_t)^{k_j} \mid \mathcal{F}_s\right] = \prod_{j=1}^d (S^j_s)^{k_j} \exp\left(\kappa \frac{t-s}{2}\right),$$

with $\kappa = \left(\left|\sum_{j=1}^d k_j \sigma_j\right|^2 - \sum_{j=1}^d k_j |\sigma_j|^2\right)$. We observe that S is indeed a polynomial process.

For the SABR model, we consider the monomial $x_1^{k_1} x_2^{k_2}$, and fix some time $t \geq 0$. We make the ansatz that we can find a martingale of the form

$$M_s = \sum_{j=0}^{k_1} c_j(t-s)(S_s^1)^j (S_s^2)^{k_1+k_2-j},$$

where c_j are some deterministic, differentiable functions of time such that $c_{k_1}(0) = 1$ and $c_j(0) = 0$ for other values of j. By Itô's formula, we obtain

$$\begin{aligned} dM_s = & \sum_{j=1}^{k_1} j c_j(t-s)(S_s^1)^{j-1}(S_s^2)^{k_1+k_2-j+1} dW_t + \sum_{j=0}^{k_1} \alpha(k_1+k_2-j)c_j(t-s)(S_s^1)^j(S_s^2)^{k_1+k_2-j} dB_t \\ & - \sum_{j=0}^{k_1} c_j'(t-s)(S_s^1)^j(S_s^2)^{k_1+k_2-j} ds + \frac{1}{2}\sum_{j=2}^{k_1} j(j-1)c_j(t-s)(S_s^1)^{j-2}(S_s^2)^{k_1+k_2-j+2} ds \\ & + \frac{1}{2}\sum_{j=0}^{k_1} \alpha^2(k_1+k_2-j)(k_1+k_2-j-1)c_j(t-s)(S_s^1)^j(S_s^2)^{k_1+k_2-j} ds \\ & + \rho \sum_{j=1}^{k_1} \alpha j(k_1+k_2-j)c_j(t-s)(S_s^1)^{j-1}(S_s^2)^{k_1+k_2-j+1} ds. \end{aligned}$$

Collecting the coefficients, we obtain the following system of ODEs:

$$c_j'(u) = \frac{(j+2)(j+1)}{2} c_{j+2}(u) + \frac{\alpha^2(k_1+k_2-j)(k_1+k_2-j-1)}{2} c_j(u) + \rho\alpha(j+1)(k_1+k_2-j-1)c_{j+1}(u),$$

where we set $c_{k_1+1} = c_{k_1+2} = 0$. Letting $\mathbf{c} = (c_0, ..., c_{k_1})^T$, we can write this equation in the form

$$\mathbf{c}' = B\mathbf{c}$$

for a certain matrix B with the coefficients of the equation, and therefore we obtain

$$\mathbf{c}(t) = \exp(tB)\mathbf{c}_0$$

where $c_0 = (0, 0, ..., 0, 1)^T$. In conclusion, we obtain that M_s is a local martingale, where the coefficients $\mathbf{c}$ are given above. One can show that the moments of S are finite, which implies that M is even a true martingale. Therefore, we obtain that

$$E[(S_t^1)^{k_1}(S_t^2)^{k_2} \mid \mathcal{F}_s] = E[M_t \mid \mathcal{F}_s] = M_s = \sum_{j=0}^{k_1} c_j(t-s)(S_s^1)^j(S_s^2)^{k_1+k_2-j}.$$

Thus, S is a polynomial model.

Exercise 11.3 Find the delta hedge for a payoff of the form $p(S_t)$, for p a polynomial.

Solution 11.3 For the Black-Scholes model, we use Itô's formula to differentiate

$$\begin{aligned} d\left(\prod_{j=1}^{d}(S_s^j)^{k_j}\exp\left(\kappa\frac{t-s}{2}\right)\right) &= \exp\left(\kappa\frac{t-s}{2}\right)\prod_{j=1}^{d}(S_s^j)^{k_j}\sum_{j=1}^{d}k_j\sigma_j dW_s \\ &= \exp\left(\kappa\frac{t-s}{2}\right)\sum_{j=1}^{d}\left(k_j(S_s^j)^{k_j-1}\prod_{i\neq j}(S_s^i)^{k_i}\right)dS_s^j. \end{aligned}$$

Thus, the delta hedge is given by the strategy $\vartheta=(\vartheta^1,...,\vartheta^d)^T$, with

$$\vartheta_s^j = k_j\exp\left(\kappa\frac{t-s}{2}\right)(S_s^j)^{k_j-1}\prod_{i\neq j}(S_s^i)^{k_i},$$

where $\vartheta^j=0$ if $k_j=0$. The initial value is

$$\varphi_0=\prod_{j=1}^{d}(S_0^j)^{k_j}\exp\left(\frac{\kappa t}{2}\right).$$

Likewise, for the SABR model, we obtain

$$dM_s=\left(\sum_{j=0}^{k_1}jc_j(t-s)(S_s^1)^{j-1}(S_s^2)^{k_1+k_2-j}\right)dS_s^1+\left(\sum_{j=0}^{k_1}(k_1+k_2-j)c_j(t-s)(S_s^1)^{j}(S_s^2)^{k_1+k_2-j-1}\right)dS_s^2,$$

so that the delta hedge is given by

$$\begin{aligned} \vartheta_s^1 &= \sum_{j=0}^{k_1}jc_j(t-s)(S_s^1)^{j-1}(S_s^2)^{k_1+k_2-j}, \\ \vartheta_s^2 &= \sum_{j=0}^{k_1}(k_1+k_2-j)c_j(t-s)(S_s^1)^{j}(S_s^2)^{k_1+k_2-j-1}, \end{aligned}$$

as well as $\varphi_0=M_0$.

Exercise 11.4 Note that the models under consideration are Markovian. Define and compute the transition semigroup $(P_t)_{t\geq 0}$ (under Q), as it acts on the set of real polynomials $\mathrm{Pol}(\mathbb{R}^d)$ ($\mathbb{R}^2$ in the case of the SABR model).

Show that in this setting,

$$P_{t-s}f(X_s) = (\nabla P_{t-\cdot}f(S_\cdot) \bullet S)_s + P_t(x), \quad Q^x\text{-a.s.},$$

where Q^x is the law of S started from a given point x.

Show that this equality also holds P-almost surely, where P is the historical measure.

Solution 11.4 The transition semigroup P_t acts on $\mathrm{Pol}(\mathbb{R}^d)$ by

$$P_t(f) = g,$$

where f is some polynomial and g is the unique polynomial such that

$$E[f(S_t) \mid \mathcal{F}_0] = g(S_0),$$

or in other words,

$$E_x[f(S_t)] = g(x).$$

Thanks to the calculations above, we can compute P_t for monomials (and thus all polynomials, by linearity):

$$P_t\left(\prod_{j=1}^d x_j^{k_j}\right) = \exp\left(\frac{\kappa t}{2}\right)\prod_{j=1}^d x_j^{k_j},$$

for the Black-Scholes model (note that κ depends on the coefficients k_j).

On the other hand, for the SABR model, we have

$$P_t(x_1^{k_1}x_2^{k_2}) = \sum_{j=0}^{k_1} c_j(t)(x_1)^j(x_2)^{k_1+k_2-j}.$$

From our previous considerations, in order to show the equality it is enough to check that

$$\nabla P_{t-\cdot}f(S_\cdot) = \vartheta.$$

This is a simple calculation: in the Black-Scholes model, and for f a monomial as above,

$$\begin{aligned}
\partial_j P_{t-s}f(S_s) &= \partial_j\left(\exp\left(\frac{\kappa(t-s)}{2}\right)\prod_{j=1}^d x_j^{k_j}\right)\Big|_{x=S_s} \\
&= k_j \exp\left(\frac{\kappa(t-s)}{2}\right)(S_s^j)^{k_j-1}\prod_{i\neq j} x_i^{k_i} \\
&= k_j \exp\left(\frac{\kappa(t-s)}{2}\right)(S_s^j)^{k_j-1}\prod_{i\neq j} (S_s^i)^{k_i} \\
&= \vartheta_s^j.
\end{aligned}$$

Finally, the equality also holds P-a.s., since stochastic integrals are preserved under equivalent measures, and likewise, almost sure equality is also preserved.

Exercise 11.5 (Python)

Implement the discretised delta hedge from exercise 3, for the payoff $H = (S_T)^3$. Compute the error between the hedge and the payoff, as well as the difference between the payoff and a hedging strategy that only trades in S, but not Y.