Mathematical Finance

Exercise sheet 12

Exercise 12.1 Let U be a utility function satisfying the Inada conditions, i.e. $U \in C^1(\mathbb{R}_+;\mathbb{R})$ is strictly increasing, strictly concave and

$$U'(0) := \lim_{x \searrow 0} U'(x) = +\infty$$
$$U'(+\infty) := \lim_{x \to +\infty} U'(x) = 0.$$

Let J be the Legendre transform of $-U(-\cdot)$,

$$J(y) := \sup_{x>0}(U(x) - xy),$$

and denote by $I := (U')^{-1}$ the inverse of the derivative of U.

Show the following properties:

1. J is strictly decreasing and strictly convex.
2. $J'(0) = -\infty$, $J'(+\infty) = 0$, $J(0) = U(+\infty)$ and $J(+\infty) = U(0)$.
3. For any $x > 0$,
$$U(x) = \inf_{y>0}(J(y) + xy).$$
4. For any $y > 0$,
$$J(y) = U(I(y)) - yI(y).$$
5. $J' = -I$.

Solution 12.1

First we show that the supremum defining J is a maximum, i.e. for any $y > 0$, we have

$$J(y) = \sup_{x>0}(U(x) - xy) = U(x_y) - x_y y$$

for some $x_y > 0$.

Note that, letting $g_y(x) = U(x) - xy$, we have that g_y is differentiable and

$$g_y'(x) = U'(x) - y.$$

Since U is C^1 (i.e. U' is continuous), strictly convex (i.e. U' is strictly decreasing), with $U'(0) = +\infty$ and $U'(+\infty) = 0$ by the Inada conditions, we obtain exactly one solution to $U'(x_y) = y$. Moreover, g_y' is negative for $x > x_y$ and positive for $x < x_y$. Thus the maximum is obtained exactly at $x_y = (U')^{-1}(y)$, which is a continuous, decreasing function of y (since U' is). We keep using the notation x_y.

On the other hand, we can see that for $y \leq 0$, the maximum is obtained as $x \to +\infty$ (since g_y is increasing in x), which gives that $J(0) = U(+\infty)$ (possibly $= +\infty$) and $J(y) = +\infty$ for $y < 0$.

We show first that J is differentiable on $(0, +\infty)$. Note that the equation $U'(x_y) = y$ implies that x_y is increasing in y. Thus, we have the following: picking some arbitrary $\bar{y}$, and letting $\bar{x} = x_{\bar{y}}$,

$$\begin{aligned} J(y) &= U(x_y) - x_y y \\ &= U(\bar{x}) - \bar{x}\bar{y} + \int_{\bar{x}}^{x_y} U'(w)dw - \int_{\bar{y}}^{y} (sdx_s + x_s ds) \\ &= U(\bar{x}) - \bar{x}\bar{y} + \int_{\bar{y}}^{y} U'(x_s)dx_s - \int_{\bar{y}}^{y} (sdx_s + x_s ds) \end{aligned}$$

(using a Riemann-Stieltjes integral, chain rule and integration by parts).

Since $U'(x_s) = s$, this simplifies as

$$J(y) = J(\bar{y}) - \int_{\bar{y}}^{y} x_s ds.$$

But since $x_s = (U')^{-1}(s)$ is a continuous function of s, this shows that J is differentiable with $J'(y) = -x_y$.

1. J is strictly decreasing and strictly convex since $J'(y) = -(U')^{-1}(y)$ is a strictly negative and strictly increasing function of y.

2. We have $J'(0) = -(U')^{-1}(0) = -\infty$ and $J'(+\infty) = -(U')^{-1}(+\infty) = 0$, by the Inada conditions. We already saw earlier that $J(0) = U(+\infty)$.

   Note that $J(y) = \sup_{x>0}(U(x) - xy) \geq U(0)$ for any $y > 0$, by taking $x \to 0$. Moreover, $J(y) \leq U(\epsilon)$ for small enough $y > 0$, since for $y \geq U'(\epsilon)$, we have

   $$U(x) - xy \leq U(x) \leq U(\epsilon)$$

   if $x \leq \epsilon$, and we know that the maximiser x_y must be in $[0, \epsilon]$ (since U' is decreasing). Thus, taking $\epsilon \to 0$ we get $J(+\infty) = U(0)$.

3. By definition, $J(y) = \sup_{x>0}(U(x) - xy)$, from which we see that

   $$J(y) - U(x) \geq -xy$$

   for any $y, x \geq 0$. We can also write it as

   $$U(x) \leq J(y) + xy.$$

   On the other hand, for any $x > 0$ we know that this inequality is attained at $y = U'(x)$, and therefore

   $$U(x) = \inf_{y>0} J(y) + xy.$$

   In the case of $x = 0$, we saw before that $U(0) = J(+\infty)$, and we can show that this is equal to the infimum.

4. We already showed this above.

5. Likewise.

Exercise 12.2 Let the financial market $S = (S_k)_{k=0,\dots,N}$ be defined over the *finite* filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0,\dots,N}, P)$ and satisfy $\mathcal{M}^a(S) \neq \emptyset$, and let U be a utility function satisfying the Inada conditions. Consider the value functions

$$u(x) = \sup_{X_T \in C(x)} E[U(X_T)] \text{ and } v(y) = \inf_{Q \in \mathcal{M}^a(S)} E\left[V\left(y\frac{dQ}{dP}\right)\right],$$

where V is the convex conjugate of U and

$$C(x) = \{X_T \in L^0(\Omega, \mathcal{F}_T, P) \mid \forall Q \in \mathcal{M}^a(S) : E_Q[X_T] \leq x\}.$$

Show that the optimisers $\hat{X}_T(x)$, $\hat{Q}(x)$ and $\hat{y}(x)$ satisfy $U'\left(\hat{X}_T(x)\right) = \hat{y}(x)\frac{d\hat{Q}(x)}{dP}$ for each $x \in \operatorname{dom}(U)$.

Solution 12.2 From minimax considerations, and writing the Lagrangian $L(X_T, y, Q) = E\left[U(X_T) - y\left(\frac{dQ}{dP}X_T - x\right)\right]$ for $X_T \in L^0(\Omega)$, $y > 0$ and $Q \in \mathcal{M}^a(S)$, we have

$$\begin{aligned}
\sup_{X_T \in C(x)} E[U(X_T)] &= \sup_{X_T} \inf_{y>0, Q \in \mathcal{M}^a(S)} L(X_T, y, Q) \\
&= \sup_{X_T} \inf_{y>0, Q \in \mathcal{M}^a(S)} E\left[U(X_T) - y\left(\frac{dQ}{dP}X_T - x\right)\right] \\
&= \inf_{y>0, Q \in \mathcal{M}^a(S)} \sup_{X_T} E\left[U(X_T) - y\left(\frac{dQ}{dP}X_T - x\right)\right] \\
&= \inf_{y>0, Q \in \mathcal{M}^a(S)} E\left[V\left(y\frac{dQ}{dP}\right)\right] - xy \\
&= \inf_{y>0} v(y) - xy.
\end{aligned}$$

Note that we obtain the third line by maximising for each fixed ω, since we no longer have any constraint on X_T (other than measurability). From question 1 we know that this supremum is attained exactly when $U'(X_T) = y\frac{dQ}{dP}$. Furthermore, the last infimum is attained exactly when $y = \hat{y}(x)$, and the infimum over the martingale measure is obtained exactly when $\frac{dQ}{dP} = \frac{d\hat{Q}(x)}{dP}$. Likewise, the supremum over X_T is attained exactly when $\hat{X}_T(x)$. Therefore, the unique saddle point of the Lagrangian satisfies $U'\left(\hat{X}_T(x)\right) = \hat{y}(x)\frac{d\hat{Q}(x)}{dP}$, as we wanted.

Exercise 12.3 (optional) Consider the utility function $u_\gamma(x) = \frac{x^\gamma}{\gamma}$, for $x > 0$ and $\gamma \in (-\infty, 1)\setminus\{0\}$. Show that $u_\gamma(x) - \frac{1}{\gamma} \to \log x$ as $\gamma \to 0$. Compute the conjugate functions of u_γ and log.

Solution 12.3 Fixing x, we can easily see that

$$\begin{aligned} u_\gamma(x) - \frac{1}{\gamma} &= \frac{x^\gamma - 1}{\gamma} \\ &= \frac{e^{\gamma \log x} - 1}{\gamma} \\ &= \frac{1 + \gamma \log x + o(\gamma) - 1}{\gamma} \\ &= \log x + o(1) \to \log x \end{aligned}$$

as $\gamma \to 0$. This shows the desired convergence.

To compute the conjugate functions, we have that

$$\begin{aligned} v_\gamma(u'_\gamma(x)) &= -x u'_\gamma(x) + u_\gamma(x) \\ \Rightarrow v_\gamma(x^{\gamma-1}) &= -x^\gamma + \frac{x^\gamma}{\gamma} \\ \Rightarrow v_\gamma(x^{\gamma-1}) &= \frac{1-\gamma}{\gamma} x^\gamma \\ \Rightarrow v_\gamma(y) &= \frac{\gamma - 1}{\gamma} y^{\frac{\gamma}{\gamma-1}}. \end{aligned}$$

For the logarithm,

$$\begin{aligned} v(\log'(x)) &= -x \log'(x) + \log(x) \\ \Rightarrow v\left(\frac{1}{x}\right) &= -1 + \log(x) \\ \Rightarrow v(y) &= -1 - \log(y). \end{aligned}$$

Exercise 12.4 Assume that the interest rate is 0, i.e. there exists a riskless asset with constant value 1, and consider the Bachelier model

$$dS_t = \mu dt + \sigma dB_t, \quad S_0 \in \mathbb{R},$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$.

Compute the optimal utility and optimal strategy associated with the problem

$$J_0 = \sup_{\vartheta \in \Theta^x_{\mathrm{adm}}} E\left[u\left(x + \int_0^T \vartheta_s dS_s\right)\right],$$

for the cases of power utility $u_\gamma(x)$ and log-utility $u(x) = \log(x)$.

Hint. To find a good ansatz for the log-utility case, try (heuristically) taking a limit of the power utility case as $\gamma \to 0$.

Solution 12.4 We compute the Hamilton-Jacobi-Bellman equation. Assuming that we have

$$J_t(\vartheta) = \operatorname{ess\,sup}_{\psi \in \Theta(t,\vartheta)} E[U(X^\psi_T) \mid \mathcal{F}_t] = k(t, X^\vartheta_t),$$

where $X^\vartheta = x + \vartheta \bullet S$, we can use Itô's formula to obtain

$$dJ_t(\vartheta) = k_t(t, X^\vartheta_t)dt + k_x(t, X^\vartheta_t)\vartheta_t(\mu dt + \sigma dB_t) + \frac{1}{2}k_{xx}(t, X^\vartheta_t)\vartheta_t^2\sigma^2 dt.$$

With the martingale optimality principle in mind, we maximise over p the expression

$$k_t(t, X^\vartheta_t) + \mu k_x(t, X^\vartheta_t)p + \frac{\sigma^2}{2}k_{xx}(t, X^\vartheta_t)p^2,$$

which has a maximum at

$$\hat{p} = -\frac{\mu k_x(t, X^\vartheta_t)}{\sigma^2 k_{xx}(t, X^\vartheta_t)}.$$

Setting $p = \hat{p}$, the (heuristic) drift term for the optimal strategy $\hat{\vartheta}$ becomes

$$k_t(t, X^{\hat{\vartheta}}_t) - \frac{\mu^2 k_x^2(t, X^{\hat{\vartheta}}_t)}{2\sigma^2 k_{xx}(t, X^{\hat{\vartheta}}_t)} = 0,$$

by the martingale optimality principle. We then obtain the HJB equation for k:

$$k_t - \frac{\mu^2 k_x^2}{2\sigma^2 k_{xx}} = 0.$$

Now, for power utility u_γ, we try a solution of the form

$$k(t, x) = f(t)\frac{x^\gamma}{\gamma},$$

where $f(T) = 1$ to satisfy the terminal condition. With this ansatz, we obtain

$$\begin{aligned} f'(t)\frac{x^\gamma}{\gamma} &= \frac{\mu^2 f(t)^2 x^{2\gamma-2}}{2\sigma^2(\gamma-1)f(t)x^{\gamma-2}} \\ \Rightarrow f'(t) &= \frac{\mu^2\gamma}{2\sigma^2(\gamma-1)}f(t) \\ \Rightarrow f(t) &= e^{\kappa(T-t)}, \end{aligned}$$

where $\kappa_\gamma = -\frac{\mu^2\gamma}{2\sigma^2(\gamma-1)}$.

The optimal strategy is then given by the feedback control

$$\hat{\vartheta}_t = \hat{p} = -\frac{\mu X_t^{\hat{\vartheta}}}{\sigma^2(\gamma - 1)}.$$

In the case of logarithmic utility, we can find the following ansatz by taking a limit (or otherwise):

$$k(t, x) = g(t) + \log(x)$$

where $g(T) = 0$. With this ansatz, we obtain

$$\begin{aligned} g'(t) &= \frac{\mu^2 \frac{1}{x^2}}{-2\sigma^2 \frac{1}{x^2}} \\ \Rightarrow g'(t) &= -\frac{\mu^2}{2\sigma^2} \\ \Rightarrow g(t) &= \kappa(T - t), \end{aligned}$$

where $\kappa = \frac{\mu^2}{2\sigma^2}$.

The optimal strategy is then given by the feedback control

$$\hat{\vartheta}_t = \hat{p} = \frac{\mu X^{\hat{\vartheta}_t}}{\sigma^2}.$$