Mathematical Finance

Exercise sheet 13

Exercise 13.1 Consider a complete financial market with time interval $[0,T]$, riskless asset $S_t^0 = e^{rt}$ for some $r \in \mathbb{R}$ and a d-dimensional risky asset S, with their natural filtration and unique separating measure Q.

(a) Find the arbitrage-free price at time t of a bounded European ($\mathcal{F}_T$-measurable) payoff H, denoted by $\pi_t^E(H)$.

(b) Let $(U_t)_{t\in[0,T]}$ be a non-negative bounded adapted process. Find the arbitrage-free price at time t of the American option with payoff U, denoted by $\pi_t^A(U)$.

(c) Give an alternative characterisation of $\pi_t^A(U)$ as a Snell envelope.

(d) In terms of an European option, give a necessary and sufficient condition for a given stopping time τ to be an optimal exercise time of the American option.

(e) Suppose that $r \geq 0$ and that the riskless asset follows a Black-Scholes model. Show that the American call option has the same value as the European call option.

(f) Suppose that U is continuous and uniformly bounded. Define the stopping time $\tau = \inf\{t \geq 0 : \pi_t^A(U) = U_t\}$. Show that $\tau \leq T$, that τ is an optimal exercise time for the American option and that the stopped process $(\pi^A/S^0)^\tau$ is a Q-martingale.

(g) Suppose $r = 0$. Let M be a non-negative local martingale such that $M_0 = 1$ and $M_t = 0$ for all $t \geq 1$, and that M^t is a martingale for each $t \in [0,1)$. Consider the process $U_t = M_t + t$ on $[0,1]$ (note that U is not bounded in this case). Show that $V_t = M_t + 1$ on $[0,1)$ and $V_1 = U_1 = 1$. Deduce that $\tau = 1$ is not optimal.

Solution 13.1

(a) Since the market is complete, and since H is bounded and S^0 is deterministic, the discounted arbitrage-free price is a martingale under Q. Therefore, the undiscounted price is

$$\pi_t^E(H) = S_t^0 E_Q[H/S_T^0 \mid \mathcal{F}_t].$$

(b) Likewise, the arbitrage-free price is equal to the superreplication, since the market is complete. By results from the lectures, this price is equal to

$$\pi_t^A(U) = S_t^0 \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{t,T}} E_Q[U_\tau/S_\tau^0 \mid \mathcal{F}_t].$$

(c) It holds that

$$\pi_t^A(U)/S_t^0 = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{t,T}} E_Q[U_\tau/S_\tau^0 \mid \mathcal{F}_t]$$

is a supermartingale under Q. Moreover, $\pi^A(U)/S^0 \geq U/S^0$, and one can show that $\pi^A(U)/S^0$ is the minimal supermartingale satisfying this lower bound. Therefore, $\pi^A(U)/S^0$ is the Snell envelope of U/S^0.

(d) Note that τ is optimal if and only if

$$\pi_0^A(U) = E_Q[U_\tau / S_\tau^0].$$

Using the European pricing formula, this is equivalent to saying that

$$\pi_0^A(U) = \pi_0^E(U_\tau S_T^0 / S_\tau^0).$$

Therefore, the American option is equivalent to an European option delivering a payoff U_τ at the random time τ, or equivalently the payoff $U_\tau S_T^0 / S_\tau^0$ at time T (which is adjusted for the interest rate).

(e) For any $\tau \leq T$, we have that

$$\begin{aligned}(S_\tau - K)^+ e^{-r\tau} &= (S_\tau e^{-r\tau} - K e^{-r\tau})^+ \\ &\leq (S_\tau e^{-r\tau} - K e^{-rT})^+ \\ &\leq E[(S_T e^{-rT} - K e^{-rT})^+ \mid \mathcal{F}_\tau] \\ &= \pi_0^E((S_T - K)^+),\end{aligned}$$

using Jensen's inequality, since $(x - K e^{-rT})^+$ is a convex function and $S e^{-r\cdot}$ is a martingale under Q.

(f) Note that $\pi_T^A(U) = U_T$, therefore $\tau \leq T$.

We show that τ is optimal. Define $\tau_n := \inf\{t \geq 0 : \pi_t^A(U) \leq U_t + 1/n\} \wedge T$. We claim that $\tau = \hat{\tau} := \lim_{n\to\infty} \tau_n$. Indeed, (τ_n) is an increasing sequence, so that it increases to some limit $\hat{\tau}$, and each $\tau_n \leq \tau$ so that $\hat{\tau} \leq \tau$. If ω is such that (τ_n) converges stationarily, i.e. $\tau_m(\omega) = \tau_{m+1}(\omega) = \ldots = \hat{\tau}(\omega)$ for some m, then it is clear that $\hat{\tau}(\omega) = \tau(\omega)$, since for each n one can find t_n such that $t_n \leq \hat{\tau} + \frac{1}{n}$ and $\pi_{t_n}^A(U) \leq U_{t_n} + 1/n$, and since $\pi^A(U)$ and U are càdlàg, it follows that $\pi_{\hat{\tau}}^A(U) = U_{\hat{\tau}}$.

Consider the case where $\tau_n(\omega)$ is strictly increasing to $\hat{\tau}(\omega)$. Note that, for each n,

$$E[V_{\hat{\tau}} \mid \mathcal{F}_{\tau_n}] \leq U_{\tau_n} + 1/n,$$

by definition of τ_n and the supermartingale property. The right-hand side converges to $U_{\hat{\tau}}$ as $n \to \infty$, by continuity of U, while the left-hand side converges to $E[V_{\hat{\tau}} \mid \mathcal{F}_{\hat{\tau}-}]$. But it holds that $V_{\hat{\tau}} \geq U_{\hat{\tau}}$ almost surely, and therefore this implies that $V_{\hat{\tau}} = U_{\hat{\tau}}$ and $\tau = \hat{\tau}$.

With this preliminary step completed, we show that τ is optimal. For each n, one can find a stopping time σ such that

$$E_Q[U_\sigma / S_\sigma^0] \geq \pi_0^A(U) - 1/n^2.$$

First, we show that one may replace σ with $\sigma \wedge \tau$. Indeed, we have

$$\begin{aligned}E_Q[U_\sigma / S_\sigma^0] &= E_Q[E_Q[U_\sigma / S_\sigma^0 \mid \mathcal{F}_\tau]] \\ &= E_Q[E_Q[\mathbb{1}_{\sigma > \tau} U_\sigma / S_\sigma^0 \mid \mathcal{F}_\tau] + E_Q[\mathbb{1}_{\sigma \leq \tau} U_\sigma / S_\sigma^0 \mid \mathcal{F}_\tau]] \\ &= E_Q[\mathbb{1}_{\sigma > \tau} E_Q[U_{\sigma \vee \tau} / S_{\sigma \vee \tau}^0 \mid \mathcal{F}_\tau] + \mathbb{1}_{\sigma \leq \tau} E_Q[U_\sigma / S_\sigma^0 \mid \mathcal{F}_\tau]] \\ &\leq E_Q[\mathbb{1}_{\sigma > \tau} V_\tau / S_\tau^0 + \mathbb{1}_{\sigma \leq \tau} E_Q[U_\sigma / S_\sigma^0 \mid \mathcal{F}_\tau]] \\ &= E_Q[\mathbb{1}_{\sigma > \tau} U_\tau / S_\tau^0 + \mathbb{1}_{\sigma \leq \tau} E_Q[U_\sigma / S_\sigma^0 \mid \mathcal{F}_\tau]] \\ &= E_Q[E_Q[\mathbb{1}_{\sigma > \tau} U_\tau / S_\tau^0 + \mathbb{1}_{\sigma \leq \tau} U_\sigma / S_\sigma^0 \mid \mathcal{F}_\tau]] \\ &= E_Q[U_{\sigma \wedge \tau} / S_{\sigma \wedge \tau}^0].\end{aligned}$$

So, we may assume that $\sigma \leq \tau$. Next, we show that $Q(\sigma < \tau_n) \leq 2/n$. Indeed, note that on $\{\sigma < \tau_n\}$, it holds that $V_\sigma \geq U_\sigma + \frac{1}{n}$. By definition, one can find a new stopping time σ' such that $\sigma' \geq \sigma$, $\sigma' = \sigma$ on $\{\sigma \geq \tau_n\}$ and $E[U_{\sigma'}/S^0_{\sigma'} \mid \mathcal{F}_\sigma] \geq U_\sigma/S^0_\sigma + \frac{1}{2n}$ on $\sigma < \tau_n$. In particular, we have

$$\begin{aligned}
\pi_0^A(U) &\geq E_Q[U_{\sigma'}/S^0_{\sigma'}] \\
&= E_Q[E_Q[U_{\sigma'}/S^0_{\sigma'} \mid \mathcal{F}_\sigma]] \\
&\geq E_Q\left[\mathbb{1}_{\sigma \geq \tau_n} U_\sigma/S^0_\sigma + \mathbb{1}_{\sigma < \tau_n}\left(U_\sigma/S^0_\sigma + \frac{1}{2n}\right)\right] \\
&= E_Q[U_\sigma/S^0_\sigma] + \frac{1}{2n} Q(\sigma < \tau_n),
\end{aligned}$$

which combined with the definition of σ gives the result.

Next, we have that

$$\begin{aligned}
\pi_0^A(U) - \frac{1}{n^2} &\leq E_Q[U_\sigma/S^0_\sigma] \\
&= E_Q[\mathbb{1}_{\sigma < \tau_n} U_\sigma/S^0_\sigma] + E_Q[\mathbb{1}_{\tau \geq \sigma \geq \tau_n} U_\sigma/S^0_\sigma] \\
&\leq \frac{2}{n} E_Q[\sup_{u \in [0,\tau_n]} V_u/S^0_u] + E_Q[\sup_{u \in [\tau_n, \tau]} |U_u/S^0_u - U_\tau/S^0_\tau|] + E_Q[U_\tau/S^0_\tau].
\end{aligned}$$

Clearly, the left-hand side tends to 0 as $n \to \infty$. Since U is bounded, which implies that V is bounded as well, the first term on the right also tends to 0 as $n \to \infty$. Moreover, the second term on the right converges to 0 as $n \to \infty$ by the dominated convergence theorem.

Therefore, we obtain that

$$\pi_0^A(U) \leq E_Q[U_\tau/S^0_\tau],$$

which shows optimality.

The martingale property then follows easily, since V/S^0 is a supermartingale, and for any $t \in [0,T]$ it holds that

$$E[V_{t\wedge\tau}/S^0_{t\wedge\tau}] \geq E[U_\tau/S^0_\tau] = E[V_0].$$

(g) It is clear that $U_1 = 1$ and $V_1 = U_1$, since $\mathcal{S}_{1,1} = \{1\}$.

Since M is a non-negative local martingale, it is a supermartingale. We have that, for $t \in [0,1)$ and any stopping time $\tau \leq 1$,

$$E[U_\tau \mid \mathcal{F}_t] = E[M_\tau + \tau \mid \mathcal{F}_t] \leq M_t + 1.$$

Therefore, $V_t \leq M_t + 1$.

Conversely, let τ_n be a localising sequence for M. Then, we have that

$$V_t \geq E[U_{\tau_n \wedge 1} \mid \mathcal{F}_t] = E[M_{\tau_n \wedge 1} + \tau_n \wedge 1 \mid \mathcal{F}_t] = M_{\tau_n \wedge t} + E[\tau_n \wedge 1 \mid \mathcal{F}_t].$$

Since $\tau_n \uparrow \infty$ almost surely, this converges to $M_t + 1$ almost surely.

Finally, we conclude that $\tau = 1$ is not optimal. Indeed, $E[U_\tau] = 1$, but $V_0 = M_0 + 1 = 2$, so that τ does not achieve the optimal value.

Exercise 13.2 Consider a Bachelier model with riskless asset of constant price 1 and a one-dimensional risky asset of price

$$S_t = S_0 + \sigma B_t,$$

for some constant S_0.

Consider the payoff process

$$U_t = g\left(S_t, Y_t\right),$$

for g a non-negative bounded measurable function of the asset price S and its modified running average $Y_t = \frac{1}{t+1}\left(Y_0 + \int_0^t S_u du\right)$ (for some constant Y_0).

(a) Argue why the value of the American option associated with U can be expressed as

$$\pi_t^A(U) = f\left(t, S_t, Y_t\right)$$

for some function f.

(b) Assuming that f is smooth enough, write a free boundary partial differential equation for f.

(c) Suppose that g is smooth. Find a condition that characterises an optimal exercise time $\tau < T$ for U, in terms of the derivatives of g.

(d) Let $g(S_t, Y_t) = (S_t - Y_t)^2$. Compute (heuristically) the optimal exercise time τ.

Solution 13.2

(a) Note that

$$\begin{aligned}
\pi_t^A(U) &= \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{t,T}} E_Q[U_\tau \mid \mathcal{F}_t] \\
&= \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{t,T}} E_Q\left[g\left(S_\tau, Y_\tau\right) \mid \mathcal{F}_t\right] \\
&= \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{t,T}} E_Q\left[g\left(S_t + \sigma(B_\tau - B_t), \frac{t+1}{\tau+1} Y_t + \frac{\tau - t}{\tau+1} S_t + \frac{1}{\tau+1}\int_t^\tau \sigma(B_u - B_t) du\right) \mid \mathcal{F}_t\right].
\end{aligned}$$

We see that the law of $(U_s)_{s \geq t}$ conditional on $\mathcal{F}_t$ depends only on S_t and Y_t (as well as t), since the increments $B_u - B_t$ are independent of $\mathcal{F}_t$. Therefore, one can expect that $\pi^A(U)$ can be expressed by such a function f.

(b) Note that $\pi^A(U)$ is a supermartingale, and moreover one can show that its finite variation is constant on $\{\pi^A(U) > U\}$. By Itô's formula we have

$$df\left(t, S_t, Y_t\right) = \partial_t f\left(t, S_t, Y_t\right) dt + \sigma \partial_s f\left(t, S_t, Y_t\right) dB_t + \frac{1}{t+1}(S_t - Y_t)\partial_y f\left(t, S_t, Y_t\right) dt + \frac{\sigma^2}{2}\partial_{ss} f\left(t, S_t, Y_t\right) dt.$$

This leads to the free boundary partial differential equation

$$\begin{cases}
\partial_t f + \frac{1}{t+1}(s-y)\partial_y f + \frac{\sigma^2}{2}\partial_{ss} f = 0, \quad \text{if } f(t,s,y) > g(s,y), \\
f(t,s,y) \geq g(s,y), \\
\partial_t f + \frac{1}{t+1}(s-y)\partial_y f + \frac{\sigma^2}{2}\partial_{ss} f \leq 0, \\
f(T,s,y) = g(s,y).
\end{cases}$$

(c) Suppose that $f(t,s,y) = g(s,y)$ for some particular t,s,y. Since $f \geq g$ on a neighbourhood of (t,s,y), we obtain that $\partial_t f(t,s,y) = 0$, as well as $\partial_y f(t,s,y) = \partial_y g(s,y)$ and $\partial_{ss} f \geq \partial_{ss} g$.

Therefore, the free boundary partial differential equation yields that

$$0 \geq \partial_t f(t,s,y) + \frac{1}{t+1}(s-y)\partial_y f(t,s,y) + \frac{\sigma^2}{2}\partial_{ss} f(t,s,y) \geq \frac{1}{t+1}(s-y)\partial_y g(t,s,y) + \frac{\sigma^2}{2}\partial_{ss} g(t,s,y).$$

In other words, if $\tau < T$ is an optimal exercise time then it must satisfy

$$\frac{\sigma^2}{2}\partial_{ss} g(\tau, S_\tau, Y_\tau) \leq -\frac{1}{\tau+1}(S_\tau - Y_\tau)\partial_y g(\tau, S_\tau, Y_\tau)$$

almost surely.

(d) From the previous part, if the optimal exercise time $\tau < T$, then we have

$$\frac{\sigma^2}{2}\partial_{ss} g(\tau, S_\tau, Y_\tau) \leq -\frac{1}{\tau+1}(S_\tau - Y_\tau)\partial_y g(\tau, S_\tau, Y_\tau).$$

Computing these derivatives, we obtain

$$\sigma^2 \leq \frac{2}{\tau+1}(S_\tau - Y_\tau)^2.$$

Assuming that the option is exercised as soon as this condition holds, we obtain that

$$\tau = \inf\{t \geq 0 : |S_t - Y_t| = \sigma\sqrt{(t+1)/2}\} \wedge T.$$

Exercise 13.3 Consider a market model where the n-dimensional stock price process X satisfies

$$d\log X_i(t) = \gamma_i(t)dt + \sum_{\nu=1}^{n} \sigma_{i\nu}(t)dW_\nu(t),$$

where the γ_i and $\sigma_{i\nu}$ are progressively measurable processes satisfying appropriate integrability conditions.

(a) Define the market portfolio μ and what it means for a portfolio π to be functionally generated by a function S.

(b) Write down the formula for the portfolio π generated by S, a positive C^2 function defined on a neighbourhood U of the simplex Δ^n such that for each i, $x_i D_i \log S(x)$ is bounded on Δ^n.

(c) Compute the portfolios generated by the following functions:

- $S(x) = 1$.
- $S(x) = w_1x_1 + \ldots + w_nx_n$, where the w_i are non-negative and not all equal to 0.
- $S(x) = x_1^{p_1} \ldots x_n^{p_n}$, where the p_i are constants adding up to 1.
- $S(x) = \left(w_1x_1^p + \ldots + w_nx_n^p\right)^{1/p}$, where the w_i are non-negative and not all equal to 0 and $p > 0$.

Solution 13.3

(a) The market portfolio has total value

$$Z_\mu(t) = X_1(t) + \ldots + X_n(t),$$

so that the weights are proportional to the market capitalisations:

$$\mu_i(t) = \frac{X_i(t)}{Z_\mu(t)}.$$

We say that a portfolio π (with value Z_π) is functionally generated by S if its relative return is given by

$$d\log(Z_\pi(t)/Z_\mu(t)) = d\log S(\mu(t)) + d\Theta(t),$$

where Θ is a finite variation process, called the drift process associated with S.

(b) For such S, the portfolio generated by S is given by the formula

$$\pi_i(t) = \left(D_i \log S(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log S(\mu(t))\right)\mu_i(t).$$

(c)
- We immediately obtain that $\pi = \mu$.
- We compute

$$\begin{aligned}
\pi_i(t) &= \left(D_i \log S(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log S(\mu(t))\right)\mu_i(t) \\
&= \left(\frac{w_i}{w_1\mu_1(t) + \ldots + w_n\mu_n(t)} + 1 - \sum_{j=1}^{n} \mu_j(t)\frac{w_j}{w_1\mu_1(t) + \ldots + w_n\mu_n(t)}\right)\mu_i(t) \\
&= \frac{w_i\mu_i(t)}{w_1\mu_1(t) + \ldots + w_n\mu_n(t)}.
\end{aligned}$$

This can be seen as a modification of the market portfolio, weighted by the parameters w_i.

- We have

$$\begin{aligned}\pi_i(t) &= \left(D_i \log S(\mu(t)) + 1 - \sum_{j=1}^n \mu_j(t) D_j \log S(\mu(t)) \right) \mu_i(t) \\ &= \left(p_i/\mu_i(t) + 1 - \sum_{j=1}^n \mu_j(t) p_j/\mu_j(t) \right) \mu_i(t) \\ &= p_i.\end{aligned}$$

This portfolio invests a fixed proportion p_i of its value in each stock i. In the case $p_1 = ... = p_n = \frac{1}{n}$, this corresponds to an equal-weighted portfolio.

- We compute

$$\begin{aligned}\pi_i(t) &= \left(D_i \log S(\mu(t)) + 1 - \sum_{j=1}^n \mu_j(t) D_j \log S(\mu(t)) \right) \mu_i(t) \\ &= \left(\frac{w_i \mu_i(t)^{p-1} \left(w_1\mu_1(t)^p + \ldots + w_n\mu_n(t)^p\right)^{-1+1/p}}{\left(w_1\mu_1(t)^p + \ldots + w_n\mu_n(t)^p\right)^{1/p}} + 1 \right. \\ &\qquad \left. - \sum_{j=1}^n \mu_j(t) \frac{w_j \mu_j(t)^{p-1} \left(w_1\mu_1(t)^p + \ldots + w_n\mu_n(t)^p\right)^{-1+1/p}}{\left(w_1\mu_1(t)^p + \ldots + w_n\mu_n(t)^p\right)^{1/p}} \right) \mu_i(t) \\ &= \frac{w_i \mu_i(t)^p}{\left(w_1\mu_1(t)^p + \ldots + w_n\mu_n(t)^p\right)}\end{aligned}$$