Mathematical Finance

Exercise sheet 4

Exercise 4.1

(a) Let W be a Brownian motion and τ an independent random variable taking non-negative real values. Consider the process

$$X = \mathcal{E}(W)^{\tau}.$$

Show that there exists a suitable choice of τ such that X is a uniformly integrable martingale but X^*_∞ is not integrable.

(b) Let $T \in (0,\infty)$ be the time horizon, L^∞ denote the class of all bounded martingales and H^∞ the class of martingales M such that $[M]_T$ is bounded. Show that $L^\infty \nsubseteq H^\infty$ and $H^\infty \nsubseteq L^\infty$.

(c) For a martingale M on $[0,T]$, denote

$$\|M\|_{BMO_2} := \sup_t \|E[|M_T - M_{t-}|^2 \mid \mathcal{F}_t]^{1/2}\|_\infty.$$

Let BMO be the set of martingales such that $\|M\|_{BMO_2} < \infty$. Show that $L^\infty, H^\infty \subseteq BMO$.

(d) Let H^1 denote the class of martingales with integrable maximum. Show that for $M \in H^1$ and $N \in BMO$, and assuming that M and N are continuous,

$$E\left[\int_0^T |d\langle M,N\rangle_s|\right] \le c\|M\|_{H_1}\|N\|_{BMO_2}.$$

Solution 4.1

(a) Clearly, for any choice of τ, wee have that X is a non-negative local martingale, hence a supermartingale. So it suffices to show that $E[X_t] = 1$ for all $t \in [0,+\infty]$ (note that $X_\infty = \mathcal{E}(W)_\tau$ is well-defined, since $\tau < \infty$ almost surely).

Indeed, letting F_τ be the cdf of τ, we have by independence

$$\begin{aligned} E[X_t] &= \int_0^\infty E[\mathcal{E}(W)_{t\wedge s}]dF_\tau(s) \\ &= \int_0^\infty dF_\tau(s) \\ &= 1. \end{aligned}$$

Thus, X is a uniformly integrable martingale. On the other hand, we note that X^*_∞ is not integrable for a suitable choice of τ. Note that $E[\mathcal{E}(M)^*_\infty] = \infty$, since otherwise $\mathcal{E}(M)$ would be a uniformly integrable martingale, which is not the case. By the monotone convergence theorem, we have that $E[\mathcal{E}(M)^*_t] \to \infty$ as $t \to \infty$. It is then clear that

$$\begin{aligned} E[X^*_\infty] &= \int_0^\infty E[\mathcal{E}(W)^*_s]dF_\tau(s) \\ &= \infty, \end{aligned}$$

for some choice of distribution F_τ.

(b) Let W be a Brownian motion. Then W^t, stopped at a deterministic time, belongs to H^∞ but not L^∞.

Conversely, W^τ where $\tau = \inf\{t \geq 0 : |W_t| \geq 1\}$ belongs to L^∞ but not H^∞, since τ is not bounded. By a time change, one can fit this process to a bounded time interval $[0, T]$.

(c) It is evident that

$$\begin{aligned} \|M\|_{BMO_2} &= \sup_t \|E[|M_T - M_{t-}|^2 \mid \mathcal{F}_t]^{1/2}\|_\infty \\ &\leq \sqrt{2}C < \infty \end{aligned}$$

if M is bounded by C.

Moreover, if $M \in H^\infty$ then M is a square-integrable martingale, so that

$$E[|M_T - M_{t-}|^2 \mid \mathcal{F}_t] = E[[M]_T - [M]_{t-}|^2 \mid \mathcal{F}_t] < C^2$$

and $M \in BMO$.

(d) We use the Kunita-Watanabe inequality to write

$$\begin{aligned} E\left[\left(\int_0^T |d\langle M, N\rangle_s|\right)\right]^2 &\leq E\left[\int_0^T \langle M_s\rangle^{-1/2} d\langle M\rangle_s\right]\left[\int_0^T \langle M_s\rangle^{1/2} d\langle N\rangle_s\right] \\ &\leq E\left[2\int_0^T d(\langle M\rangle_s^{1/2})\right] E\left[\langle N\rangle_T \langle M\rangle_T^{1/2} - \int_0^T \langle N\rangle_s d(\langle M_s\rangle^{1/2})\right] \\ &\leq 2E\left[\langle M\rangle_T^{1/2}\right] E\left[\int_0^T (\langle N\rangle_T - \langle N\rangle_s) d(\langle M_s\rangle^{1/2})\right] \\ &\leq 2E\left[\langle M\rangle_T^{1/2}\right]^2 \|N\|_{BMO_2}^2 \\ &\leq c\|M\|_{H_1}\|N\|_{BMO_2}. \end{aligned}$$

Exercise 4.2 Let B be a Brownian motion on $\mathbb{R}$ (starting at 0). For $x \in [-1,1]$, we consider $B_t^x = x + B_t$, a Brownian motion "started at x". Let $\tau^x := \inf\{t > 0 : |B_t^x| \geq 1\}$ be the first time that it exits $[-1,1]$.

(a) Let g be a continuous function on $[-1,1]$. Show that the function $u : [-1,1] \to \mathbb{R}$ defined by

$$u(x) = E\left[\int_0^{\tau_x} g(B_s^x)ds\right]$$

is well-defined and continuous.

Hint: Start by showing that τ_x is integrable by considering the martingale $(B^x)_t^2 - t$.

(b) Suppose that v is a bounded function on $[-1,1]$ such that $v(-1) = v(1) = 0$, and furthermore the process M^x defined by

$$M_t^x = v(B_{t\wedge\tau^x}^x) + \int_0^{t\wedge\tau^x} g(B_s^x)ds$$

is a local martingale for each x.

Prove that $u = v$.

(c) Suppose that v is a bounded function on $[-1,1]$ such that $v(-1) = v(1) = 0$ and it satisfies the second-order differential equation

$$\frac{1}{2}v''(x) = -g(x). \tag{1}$$

Show that $v = u$.

(d) Replacing g by the Dirac delta mass δ_y at some point $y \in \mathbb{R}$, formally compute the solution v_y to (1). The function $v_y(x) =: G(x,y)$ is called the Green's function. Can you find a solution to (1) for more general g, in terms of G?

Solution 4.2

(a) To show that τ_x is integrable, note that $B_{\tau_x}^x$ is a bounded martingale, and in particular uniformly integrable, so that $B_{\tau_x\wedge t}^x \to B_{\tau_x}^x$ almost surely and in L^1 as $t \to \infty$. Moreover, it is easy to see that B_{τ_x} takes values 1 or -1 almost surely. Next, we note that $(B_{\tau_x\wedge\cdot}^x)^2 - \tau_x \wedge \cdot$ defines a martingale, and thus we obtain that for each $t \geq 0$,

$$E[(B_{\tau_x\wedge\cdot}^x)^2 - \tau_x \wedge t] = E[(B_0^x)^2 - 0] = x^2.$$

But then, taking $t \to \infty$, the first term converges by the dominated convergence theorem (DCT) and the second converges by the monotone convergence theorem to $E[\tau_x]$, which we find to be finite.

Now, since g is continuous on $[-1,1]$ (hence bounded), it is clear that u is well-defined, as we can bound the integrand by $K\tau_x$ (where K is a bound for g), which is integrable since τ_x is.

Continuity follows from the continuity of g, as well as the continuity of B^x in x, as long as we can uniformly bound the family $\{\tau_x\}$. One approach is as follows: letting $T := \inf\{t > 0 : |B_t| \geq 2\}$, it is clear that T dominates the whole family $\{\tau_x\}$, and by similar arguments as before T is itself integrable. Therefore, continuity of u follows by DCT.

(b) Since v and g are bounded, we have that $|M_t^x| \leq a + b\tau^x$ for some constants $a, b > 0$ and all $t \geq 0, x \in [-1,1]$. Therefore, M^x is uniformly integrable. In particular, it converges a.s. and in L^1 to

$$M_\infty^x = v(B_{\tau^x}^x) + \int_0^{\tau^x} g(B_s^x)ds,$$

and therefore we have that

$$\begin{aligned} v(x) &= M_0^x \\ &= E[M_\infty^x] \\ &= E\left[\int_0^{\tau^x} g(B_s^x) ds\right] \\ &= u(x). \end{aligned}$$

(c) We use the previous part. By Itô's formula (as v is C^2, due to the differential equation) we obtain that

$$\begin{aligned} M_t^x &= v(B_{t\wedge\tau^x}^x) + \int_0^{t\wedge\tau^x} g(B_s^x) ds \\ &= v(x) + \int_0^t v'(B_{s\wedge\tau^x}^x) dB_{s\wedge\tau^x}^x + \frac{1}{2}\int_0^t v''(B_{s\wedge\tau^x}^x) d(s\wedge\tau^x) - \int_0^t g(B_s^x) d(s\wedge\tau^x) \\ &= v(x) + \int_0^t v'(B_{s\wedge\tau^x}^x) dB_{s\wedge\tau^x}^x, \end{aligned}$$

since the last two terms cancel by assumption. It is now clear that M^x is a local martingale, indeed even a martingale: since v' is continuous (hence bounded), we obtain that

$$E\left[\int_0^t v'(B_{s\wedge\tau^x}^x)^2 d(s\wedge\tau^x)\right] < \infty.$$

Thus, we can apply the previous part to get the result.

(d) Heuristically, we can integrate twice to obtain

$$\begin{aligned} v_y(x) &= v_y(-1) + \int_{-1}^x v_y'(r) dr \\ &= \int_{-1}^x \left(v_y'(-1) + \int_{-1}^r v_y''(u) du\right) dr \\ &= (x+1)v_y'(-1) + \int_{-1}^x -2\mathbb{1}_{r\geq y} dr \\ &= (x+1)v_y'(-1) + \int_y^x -2 dr \\ &= (x+1)v_y'(-1) + 2(y-x)\mathbb{1}_{x>y}. \end{aligned}$$

We already used the boundary condition $v_y(-1) = 0$, and the other condition $v_y(1) = 0$ gives that

$$2v_y'(-1) + 2(y-1) = 0 \Rightarrow v'(-1) = 1 - y.$$

We obtain

$$\begin{aligned} v_y(x) &= (x+1)(1-y) + 2(y-x)\mathbb{1}_{x>y} \\ &= (x+1)(1-y)\mathbb{1}_{x\leq y} + (1-x)(1+y)\mathbb{1}_{x>y}. \end{aligned}$$

Indeed, we can easily check that this satisfies the equation.

For the general case, note that we can formally write

$$\frac{1}{2}v''(x) = -g(x) = \int_{-1}^{1} g(y)(-\delta_y)dy,$$

and therefore, by linearity, we might hope that the solution is given by

$$v(x) = \int_{-1}^{1} g(y)v_y(x)dy.$$

Indeed this is the case, since

$$\begin{aligned} v(x) &= \int_{-1}^{1} g(y)((x+1)(1-y)\mathbb{1}_{x\leq y} + (1-x)(1+y)\mathbb{1}_{x>y})dy \\ &= (1+x)\int_{-1}^{x} g(y)(1-y)dy + (1-x)\int_{x}^{1}(1+y)g(y)dy \end{aligned}$$

and therefore

$$\begin{aligned} v''(x) &= \frac{d^2}{dx^2}\left((1+x)\int_{-1}^{x} g(y)(1-y)dy + (1-x)\int_{x}^{1}(1+y)g(y)dy\right) \\ &= \frac{d}{dx}\left(\int_{-1}^{x} g(y)(1-y)dy + (1+x)g(x)(1-x) - \int_{x}^{1}(1+y)g(y)dy - (1-x)(1+x)g(x)\right) \\ &= \frac{d}{dx}\left(\int_{-1}^{x} g(y)(1-y)dy - \int_{x}^{1}(1+y)g(y)dy\right) \\ &= g(x)(1-x) + (1+x)g(x) \\ &= 2g(x). \end{aligned}$$

Thus, we have the nice form for the solution

$$v(x) = \int_{-1}^{1} g(y)G(x,y)dy.$$

Exercise 4.3

(a) Let σ be a continuous positive function on $\mathbb{R}$, satisfying the linear growth condition:

$$|\sigma(x)| \leq K(1 + |x|)$$

for some $K > 0$. Suppose that we have a Brownian motion B and a family of processes X^x (for $x \in \mathbb{R}$) such that, for each $x \in \mathbb{R}$, the following stochastic differential equation is satisfied for all $t \geq 0$:

$$X_t^x = x + \int_0^t \sigma(X_s^x) dB_s.$$

Prove that for each time $T > 0$ there is a constant c (depending only on T, K and p but not on x) such that

$$E[((X_T^x)^*)^p] \leq c(1 + |x|^p).$$

(b) Construct a pair (X, B), where B is a Brownian motion, such that the following stochastic differential equation is satisfied:

$$X_t = \int_0^t \operatorname{sgn}(X_s) dB_s,$$

where $\operatorname{sgn}(x) = -\mathbb{1}_{x \leq 0} + \mathbb{1}_{x > 0}$.

Solution 4.3

(a) If the equation is satisfied, each X^x is a continuous local martingale. We can use Burkholder-Davis-Gundy to get the inequality

$$\begin{aligned} E[((X_t^x)^*)^p] &\leq 2^p \left(|x|^p + E\left[\sup_{0 \leq s \leq t} \left| \int_0^t \sigma(X_s^x) dB_x \right|^p \right] \right) \\ &\leq 2^p \left(|x|^p + C_p E\left[\left(\int_0^t \sigma(X_s^x)^2 ds \right)^{\frac{p}{2}} \right] \right). \end{aligned}$$

Now, by Hölder's inequality we have

$$\begin{aligned} \left(\int_0^t \sigma(X_s^x)^2 ds \right)^{\frac{p}{2}} &\leq \left[\left(\int_0^t \sigma(X_s^x)^p ds \right)^{\frac{2}{p}} \left(\int_0^t 1^{\frac{p-2}{p}} ds \right)^{\frac{p-2}{p}} \right]^{\frac{p}{2}} \\ &\leq t^{\frac{p-2}{2}} \left(\int_0^t \sigma(X_s^x)^p ds \right). \end{aligned}$$

Therefore we can bound

$$
\begin{aligned}
E[((X_t^x)^*)^p] &\le 2^p \left(|x|^p + C_p E\left[t^{\frac{p-2}{2}} \left(\int_0^t \sigma(X_s^x)^p ds \right) \right] \right) \\
&\le 2^p \left(|x|^p + C_p t^{\frac{p-2}{2}} E\left[\left(\int_0^t K^p (1+|X_s^x|)^p ds \right) \right] \right) \\
&\le 2^p \left(|x|^p + C_p t^{\frac{p-2}{2}} K^p 2^p \left(t + \int_0^t E[|X_s^x|^p] ds \right) \right) \\
&\le 2^p \left(|x|^p + C_p T^{\frac{p-2}{2}} K^p 2^p \left(T + \int_0^t E[((X_s^x)^*)^p] ds \right) \right).
\end{aligned}
$$

So, if $u_x(t) = E[((X_t^x)^*)^p]$, we have an inequality of the form

$$u_x(t) \le a \left(1 + |x|^p + \int_0^t u_x(s) ds \right)$$

for all $t \in [0, T]$ and a constant $a > 0$ independent of x. It follows by Grönwall's inequality that

$$u_x(t) \le a(1 + |x|^p) e^{at},$$

and since we consider $t = T$ we can repackage that as

$$u_x(T) \le c(1 + |x|^p)$$

for some constant $c > 0$.

(b) Take X to be a Brownian motion and define B by

$$B_t = \int_0^t \operatorname{sgn}(X_s) dX_s.$$

Note that this is well-defined, since $\operatorname{sgn}(X_.)$ is bounded and predictable. To see that last fact, note that we can find continuous bounded functions f_n converging pointwise to sgn pointwise (easiest to see with a picture), and then the $f_n(X_.)$ are continuous processes converging to $\operatorname{sgn}(X_.)$.

Moreover, we note that B is a Brownian motion. To see this, note that

$$\int_0^t \operatorname{sgn}(X_s)^2 d[X_s] = \int_0^t ds = t,$$

which both shows that B is a local martingale (since X is) and that its quadratic variation is t. We also obtain that B is continuous (since X is) and starts at 0, therefore Lévy's characterisation gives that B is a Brownian motion as well.

As for the stochastic differential equation, note that

$$
\begin{aligned}
\operatorname{sgn}(X_.) \bullet B &= \operatorname{sgn}(X_.) \bullet (\operatorname{sgn}(X_.) \bullet X) \\
&= \operatorname{sgn}(X_.)^2 \bullet X \\
&= 1 \bullet X \\
&= X,
\end{aligned}
$$

using associativity.

Exercise 4.4 (Python) Simulate a random walk $(M_n)_{n\in\mathbb{N}}$ up to time 1000, starting from 0 and with the same probability $\frac{1}{2}$ of jumping up or down (by 1) at each step.

Quoting from [1], give explicit predictable integrands g and h and constants $c_p, C_p > 0$ such that the inequalities

$$(h \bullet M)_n + c_p[M,M]_n^{\frac{3}{2}} \leq (|M|_n^*)^3 \leq C_p[M,M]_n^{\frac{3}{2}} + (g \bullet M)_n$$

hold.

Compute the values taken by these processes along your simulated random walk, and plot them together with the process M_n^3.

References

[1] Beiglböck, Mathias; Siorpaes, Pietro. *Pathwise versions of the Burkholder–Davis–Gundy inequality.* Bernoulli 21 (2015), no. 1, 360–373. doi:10.3150/13-BEJ570. https://projecteuclid.org/euclid.bj/1426597073