Mathematical Finance

Exercise sheet 5

Exercise 5.1

(a) Let ϕ be a continuous process and M a continuous local martingale started at 0. Prove that

$$(\phi \bullet M)_\cdot = \int_0^\cdot \phi_s dM_s$$

is a local martingale started at 0.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic base, such that there are two independent stopping times T and U defined on it, which are independent and have an exponential distribution $\text{Exp}(1)$. Define M by

$$M_t = \begin{cases} 0, & t < T \wedge U, \\ 1, & t \geq T \wedge U = T, \\ -1, & t \geq T \wedge U = U. \end{cases}$$

In other words, M starts at 0 and jumps once one of the stopping times arrive, with the jump being either to 1 or -1 depending on which stopping time arrives first. In the (probability 0) event that the two arrive simultaneously, M can be defined arbitrarily, i.e. we can take it to stay at 0.

Prove that M is a martingale with respect to its natural filtration $\mathcal{F}^M$. Prove that, for $\phi_t = \frac{1}{t}\mathbb{1}_{t>0}$ (which is predictable), $\phi \bullet M$ is not a local martingale with respect to $\mathcal{F}^M$.

Solution 5.1

(a) We can choose stopping times

$$\tau_n^M = \inf\{t \geq 0 : |M_t| \geq n\},$$

$$\tau_n^{[M]} = \inf\{t \geq 0 : [M]_t \geq n\},$$

$$\tau_n^\phi = \inf\{t \geq 0 : |\phi_t| \geq n\}$$

which all converge to ∞ a.s. as n does, by continuity of M, $[M]$ and ϕ respectively. It follows that $\tau_n = \tau_n^M \wedge \tau_n^{[M]} \wedge \tau_n^\phi$ are also stopping times converging to ∞ almost surely. By continuity, $M_{\cdot\wedge\tau_n}$, $[M]_{\cdot\wedge\tau_n}$ and $\phi_{\cdot\wedge\tau_n}$ are all bounded by n. In particular, $M_{\cdot\wedge\tau_n}$ is uniformly integrable, and thus a martingale.

Now, consider the stopped process

$$(\phi \bullet M)_{\cdot\wedge\tau_n}.$$

We then have that

$$E\left[\int_0^{\tau_n} \phi(s)^2 d[M]_s\right] \leq n^3$$

by the bounds imposed on ϕ and $[M]$ by the stopping time. Therefore, $(\phi \bullet M)_{\cdot\wedge\tau_n}$ is a martingale for each n, and so $(\phi \bullet M)_\cdot$ is a local martingale as we wanted.

(b) M is adapted to its natural filtration by definition, and it is integrable as it is bounded. For the martingale property, note that in the case that $T \wedge U \leq s$, we have that $M_t = M_s$ almost surely for $t \geq s$ and in particular $E[M_t \mid \mathcal{F}_s^M] = M_s$. Otherwise, $M_s = 0$ and we have

$$\begin{aligned} E[M_t \mid \mathcal{F}_s] &= E[\mathbb{1}_{t \geq T \wedge U = T} - \mathbb{1}_{t \geq T \wedge U = U} \mid \mathcal{F}_s] \\ &= P(t \geq T \wedge U = T \mid \mathcal{F}_s) - P(t \geq T \wedge U = U \mid \mathcal{F}_s) \\ &= 0 \end{aligned}$$

by symmetry between T and U.

For the second part, we want to show that $\phi \bullet M$ is not a local martingale. Let τ be an $\mathcal{F}^M$-stopping time, and assume that it is not identically 0. The key observation is that, since τ is an $\mathcal{F}^M$-stopping time, it must be constant on the event $\{\tau < T \wedge U\}$, since no information is received until the time of the first jump. From there, we can deduce that for some $\epsilon > 0$,

$$\tau \mathbb{1}_{\tau \neq 0} \geq (T \wedge U \wedge \epsilon) \mathbb{1}_{\tau \neq 0}.$$

Indeed, Ω can be split into two parts, $\{\tau \geq T \wedge U\}$ and $\{\tau < T \wedge U\}$. In the first part, the inequality is automatically satisfied. In the second, the above observation gives that $\tau = c$ for some constant $c \geq 0$. Therefore, either $c = 0$, in which case the inequality above is vacuously true, or $c > 0$ in which case we can find such an ϵ.

Given this inequality, we may compute

$$\begin{aligned} E[|M_\tau|] &= E\left[\frac{1}{T \wedge U} \mathbb{1}_{\tau \geq T \wedge U}\right] \\ &\geq E\left[\frac{1}{T \wedge U} \mathbb{1}_{T \wedge U \leq \epsilon} \mathbb{1}_{\tau \neq 0}\right] \\ &= E\left[E\left[\frac{1}{T \wedge U} \mathbb{1}_{T \wedge U \leq \epsilon} \mathbb{1}_{\tau \neq 0} \mid \mathcal{F}_0\right]\right] \\ &= E\left[\mathbb{1}_{\tau \neq 0} E\left[\frac{1}{T \wedge U} \mathbb{1}_{T \wedge U \leq \epsilon} \mid \mathcal{F}_0\right]\right] \\ &= E[\mathbb{1}_{\tau \neq 0} \cdot \infty] \\ &= \infty, \end{aligned}$$

using the fact that $\mathbb{1}_{T \wedge U \leq \epsilon} \mathbb{1}_{\tau \neq 0}$ implies $\mathbb{1}_{\tau \geq T \wedge U}$ (by our earlier analysis), that $\frac{1}{T \wedge U}$ is not integrable near 0 and, for the last line, our assumption that τ is not identically 0.

But this means that M_τ is not integrable for any stopping time that is not identically 0, and therefore M cannot be even a local martingale.

Let $\mathcal{S}$ denote the set of semimartingales and $\mathbb{S}_1 := \{H \in \mathbb{S} : ||H||_\infty \leq 1\}$ the unit ball of simple predictable processes. The Emery topology is a topology on $\mathcal{S}$ generated by the metric

$$d_E(X,Y) := \sum_{n=1}^{\infty} 2^{-n} \sup_{H \in \mathbb{S}_1} E\left[1 \wedge \sup_{t \leq n} |(H \bullet (X-Y))_t|\right].$$

Exercise 5.2 Show that

(a) $\mathcal{S}$ endowed with the Emery topology is a topological vector space.

(b) $\mathcal{S}$ is closed in the Emery topology and complete with respect to the metric d_E.

(c) Show that the Emery topology is invariant under an equivalent change of measure.

(d) Let the set of adapted càglàd processes $\mathbb{L}$ be endowed with the u.c.p. topology and the set of semimartingales $\mathcal{S}$ be endowed with the Emery topology, and let X be a semimartingale. Show that

$$J_X : \mathbb{L} \ni Y \mapsto (Y \bullet X) \in \mathcal{S}$$

is continuous.

Solution 5.2 Note that, for $(X^n) \subset \mathcal{S}$ and $X \in \mathcal{S}$, we have

$$d_E(X^n, X) \to 0$$

if and only if

$$(H^n \bullet (X^n - X)) \overset{\text{ucp}}{\to} 0 \text{ for any } (H^n) \subset \mathbb{S}_1.$$

(a) Let $X, Y \in \mathcal{S}$. We have $d_E(X+Y,0) \leq d_E(X,0) + d_E(Y,0)$ (one can see this from the corresponding triangle inequality for d), so that addition is jointly continuous.

Moreover, $d_E(cX,0) \leq d_E(X,0)$ for real $|c| \leq 1$. To show that scalar multiplication is jointly continuous, let $c^n \to c$ and $X^n \to X$, the latter in Emery topology. To show that $c_n X_n \to cX$ in Emery topology, it is enough to show that the two differences $c_n(X_n - X)$ and $(c_n - c)X$ converge to 0 in Emery topology.

The first one converges to 0 thanks to the previous observation that $d_E(c_n(X_n - X), 0) \leq d_E(X_n - X, 0) \to 0$. The second one follows from the fact that X is a good integrator, giving that

$$c^n - c \to 0 \implies (c^n - c)H^n \overset{\text{ucp}}{\to} 0 \implies (((c^n - c)H^n) \bullet X) \overset{\text{ucp}}{\to} 0$$

for any $(H^n) \subset \mathbb{S}_1$.

(b) The metric d_E is stronger than the metric d of the ucp topology. By the completeness of d for $\mathbb{D}$, a Cauchy sequence (X^n) in the metric d_E converges in d to a càdlàg process X.

- Step 1: We show that $P((H \bullet X^n)^*_T > K) \to 0$ uniformly in n and H with $||H||_\infty \leq 1$. Let $\epsilon > 0$. Since X is Cauchy, we can choose a large enough m such that $P((H \bullet (X^n - X^m))^*_T > 1) < \epsilon$ for any $H \in \mathbb{S}_1$ and $n \geq m$. Moreover, we can choose K large enough that

$$P((H \bullet X^n)^*_T > K - 1) < \epsilon$$

for any $H \in \mathbb{S}_1$ and $n = 1, ..., m$. This is possible since the X^n are good integrators and we only consider finitely many of them.

For that choice of m and K, we have that

$$P((H \bullet X^n)^*_T > K) \leq P((H \bullet X^n)^*_T > K - 1) < \epsilon$$

if $n = 1, ..., m$, and

$$P((H \bullet X^n)^*_T > K) \leq P((H \bullet X^m)^*_T > K - 1) + P((H \bullet (X^n - X^m))^*_T > 1) < 2\epsilon$$

if $n \geq m$. This shows what we wanted.

- Step 2: We show that X is a good integrator.
  Consider a simple integrand $H = \sum_{i=1}^m H_i \mathbb{1}_{(\tau_i, \tau_{i+1}]}$ for some stopping times τ_i and $\mathcal{F}_{\tau_i}$-measurable H_i bounded by 1. Then we can easily see that

$$\begin{aligned}(H \bullet Y)^*_T &\leq \sum_{i=1}^m |H_i| \sup_{s \in (\tau_i, \tau_{i+1}]} |Y_s - Y_{\tau_i}| \\ &\leq 2mY^*_T,\end{aligned}$$

  for any process Y.
  Now, let $\epsilon > 0$. Let K be large enough that $P((H \bullet X^n)^*_T > K - 1) < \epsilon$ for all $H \in \mathbb{S}_1$ and all n. Take now any $H \in \mathbb{S}_1$. If H can be decomposed into m summands as above, use the ucp convergence to find n large enough that $P((X^n - X)^*_t > \frac{1}{2m}) < \epsilon$. Then, for that choice of K (which is independent of the choice of H), we have that

$$\begin{aligned}P((H \bullet X)^*_T > K) &\leq P((H \bullet X^n)^*_T > K - 1) + P((H \bullet (X - X^n))^*_T > 1) \\ &\leq \epsilon + P(2m(X - X^n)^*_T > 1) \\ &\leq 2\epsilon.\end{aligned}$$

  Thus, X is a good integrator.
- Step 3: $X^n \to X$ in the Emery topology.
  This is now quite similar to step 2. Take $\epsilon > 0$ and $a > 0$. Find N large enough that $\sup_{H \in \mathbb{S}_1} P((H \bullet (X^m - X^n))^*_T > \frac{a}{2}) < \epsilon$, for $n, m \geq N$. For some $H \in \mathbb{S}_1$, decomposable into m summands, find $n' \geq N$ large enough that $P((X - X^{n'})^*_T > \frac{a}{4m}) < \epsilon$. Then, for that H and any $n \geq N$,

$$\begin{aligned}P((H \bullet (X - X^n))^*_T > a) &\leq P((H \bullet (X^{n'} - X^n)^*_T > \frac{a}{2}) + P((H \bullet (X - X^{n'}))^*_T > \frac{a}{2}) \\ &\leq \epsilon + P(2m(X - X^{n'})^*_T > \frac{a}{2}) \\ &\leq 2\epsilon.\end{aligned}$$

  Since the choice of N does not depend on H, this proves the result.

(c) It is clearly enough that $X^n \to 0$ in Emery metric under P if and only if the same convergence holds under Q, for any equivalent measure Q. Let Q be an equivalent measure with Radon-Nikodym derivative $\frac{dQ}{dP} = Z$, and suppose that $X^n \to 0$ in Emery metric under P. This means that for $a, T > 0$,

$$\sup_{H \in \mathbb{S}_1} P((H \bullet X^n)^*_T > a) =: \epsilon_n \to 0.$$

Now, for $H \in \mathbb{S}_1$, we have that

$$\begin{aligned}Q((H \bullet X^n)^*_T > a) &= E_P(Z \mathbb{1}_{(H \bullet X^n)^*_T > a}) \\ &\leq \sup_{A \in \Omega: P(A) \leq \epsilon_n} E_P(Z \mathbb{1}_A) =: \delta_n \to 0\end{aligned}$$

as $n \to \infty$, since $\epsilon_n \to 0$ and $\{Z\}$ is a P-uniformly integrable family (as Z is P-integrable). Since the δ_n are uniform in H, we obtain the desired convergence in Emery metric under Q. The other direction is proved by symmetry.

(d) Let $(Y^n) \subset \mathbb{L}$ such that $Y^n \overset{u.c.p.}{\to} 0$ and $(H^n) \subset \mathbb{S}_1$. Then $H^n Y^n \overset{u.c.p.}{\to} 0$ and consequently

$$(H^n \bullet (Y^n \bullet X)) = ((H^n Y^n) \bullet X) \overset{u.c.p.}{\to} 0,$$

i.e., $(Y^n \bullet X) \to 0$ in the Emery topology.

Exercise 5.3 Define fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ as a Gaussian process $(X_t)_{t\in\mathbb{R}_+}$ such $E[X_t] = 0$ for all $t \geq 0$ and the covariance function is given by

$$E[X_t X_s] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

for all $t, s \geq 0$.

We take a continuous version of X and denote it by W^H.

(a) Check that:

- The formula for the covariance is equivalent to the condition

$$E[|X_t - X_s|^2] = |t-s|^{2H}$$

for $t, s \geq 0$, together with $X_0 = 0$ almost surely.
- For $c > 0$, $(\frac{1}{c^H} W^H_{ct})_{t\geq 0}$ is a fBm of Hurst parameter H.
- For $t_0 > 0$, $(W^H_{t+t_0} - W^H_{t_0})_{t\geq 0}$ is a fBm of Hurst parameter H.
- For $H = \frac{1}{2}$, W^H is a Brownian motion.

(b) Use Birkhoff's ergodic theorem to compute the almost sure limit

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} |W^H_{k+1} - W^H_k|^p$$

for $p > 0$.

(c) Deduce that, for $H < \frac{1}{2}$, W^H has infinite quadratic variation.

Solution 5.3

(a)
- From the original formula we can deduce that $E[X_0^2] = 0$, so that it is 0 a.s. Note also that if $t = s$, we obtain $E[X_t^2] = |t|^{2H}$. Therefore,

$$\begin{aligned} E[(X_t - X_s)^2] &= E[X_t^2] + E[X_s^2] - 2E[X_t X_s] \\ &= |t|^{2H} + |s|^{2H} - (|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \\ &= |t-s|^{2H} \end{aligned}$$

as we wanted.

In the other direction, since $X_0 = 0$ a.s., we obtain that

$$E[X_t^2] = E[(X_t - X_0)^2] = |t|^{2H}$$

so that

$$\begin{aligned} |t-s|^{2H} &= E[(X_t - X_s)^2] \\ &= |t|^{2H} + |s|^{2H} - 2E[X_t X_s], \end{aligned}$$

which implies the original formula for the covariance function.

- If $Y_t = \frac{1}{c^H} W^H_{ct}$, note that $E[Y_t] = 0$, Y is continuous and
$$\begin{aligned} E[Y_t Y_s] &= E\left[\frac{1}{c^H} W^H_{ct} \frac{1}{c^H} W^H_{cs}\right] \\ &= \frac{1}{c^{2H}} \frac{1}{2}(|ct|^{2H} + |cs|^{2H} - |ct - cs|^{2H}) \\ &= \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \end{aligned}$$
so that Y is again a fBm of Hurst parameter H.
- Let $Z_t = W^H_{t+t_0} - W^H_{t_0}$ be the new process. We use the alternative characterisation from the first point: clearly Z is continuous, $E[Z_t] = 0$, $Z_0 = 0$ almost surely and
$$\begin{aligned} E[|Z_t - Z_s|^2] &= E[((W^H_{t+t_0} - W^H_{t_0}) - (W^H_{s+t_0} - W^H_{t_0}))^2] \\ &= E[(W^H_{t+t_0} - W^H_{s+t_0})^2] \\ &= |t-s|^{2H} \end{aligned}$$
so that Z is a fBM of Hurst parameter H.
- If $H = \frac{1}{2}$, we obtain that, for $t \geq s$,
$$\begin{aligned} E[W^H_t W^H_s] &= \frac{1}{2}(|t| + |s| - |t-s|) \\ &= \frac{1}{2}(t + s - (t-s)) \\ &= s. \end{aligned}$$
In general, $E[W^H_t W^H_s] = t \wedge s$. This is the covariance function of Brownian motion, and since W^H is continuous it is a Brownian motion for $H = \frac{1}{2}$.

(b) We consider the canonical space $(\Omega, \mathcal{F}, P^H)$ where $\Omega = \mathbb{R}^{\mathbb{N}}$, $\mathcal{F}$ is the cylindrical σ-algebra and P^H is the law of $(W^H_n)_{n\in\mathbb{N}}$ for W^H a fBm with parameter H. We consider the shift operator T given by $T(X_n)_{n\in\mathbb{N}} = (X_{n+1} - X_1)_{n\in\mathbb{N}}$, as well as the map f given by $f(X_n)_{n\in\mathbb{N}} = |X_1|^p$.

T is measure preserving since $W^H_{s+1} - W^H_1$ is a fBm of parameter H, and hence its values on $\mathbb{N}$ have the same joint law as those of W^H itself. Moreover, we can see that T is ergodic. Therefore, Birkhoff's ergodic theorem gives us that
$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} |W^H_{k+1} - W^H_k|^p = E[|W^H_1|^p]$$
for $p > 0$. By the definition of fBm, W^H_1 is normally distributed with distribution $\mathcal{N}(0,1)$, so that the limit is $c_p = E[|Z|^p]$ for Z a standard normal random variable.

(c) Note that, by the first part, $(2^{nH} W^H_{2^{-n}t})_{t\geq 0}$ is a fBm of Hurst parameter H. Therefore, we have the equality in law
$$\begin{aligned} \frac{1}{2^n} \sum_{k=0}^{2^n-1} |W^H_{k+1} - W^H_k|^p &\overset{\mathrm{d}}{=} \frac{1}{2^n} \sum_{k=0}^{2^n-1} 2^{nHp} |W^H_{2^{-n}(k+1)} - W^H_{2^{-n}k}|^p \\ &\overset{\mathrm{d}}{=} 2^{n(Hp-1)} \sum_{k=0}^{2^n-1} |W^H_{2^{-n}(k+1)} - W^H_{2^{-n}k}|^p. \end{aligned}$$

Thus, due to the previous part, we have the convergence at least in distribution:

$$2^{n(Hp-1)} \sum_{k=0}^{2^n-1} |W^H_{2^{-n}(k+1)} - W^H_{2^{-n}k}|^p \xrightarrow{\text{d}} c_p.$$

Since the limit in distribution is a constant, the convergence also holds in probability. In particular, if $H < \frac{1}{2}$ and $p = 2$, the term $2^{n(Hp-1)}$ goes to 0 so that the quadratic variation is infinite.

Exercise 5.4 Consider a probability space $(\Omega, \mathcal{F}, P)$, together with a d-dimensional Brownian motion $(B_t)_{t\in[0,T]}$. Consider the natural filtration $\mathcal{F}^B_t = \mathcal{F}_t$ generated by B, and suppose that $\mathcal{F}_T = \mathcal{F}$.

(a) Show that any absolutely continuous measure $Q \ll P$ has a Radon-Nikodym derivative of the form

$$\frac{dQ}{dP} = \exp\left(\int_0^T \lambda_s dB_s - \frac{1}{2}\int_0^T ||\lambda_s||^2 ds\right)$$

for some $\lambda \in L(B)$.

Hint: You may use the Itô martingale representation theorem.

(b) For Q given in the above form, and assuming that $Q \sim P$, find (with proof) a d-dimensional Brownian motion under Q.

Hint: You may use the Girsanov-Meyer theorem.

Solution 5.4

(a) Let Q be an absolutely continuous measure, with Radon-Nikodym derivative $\frac{dQ}{dP}$. Because $\frac{dQ}{dP}$ is non-negative and integrable, we can by the martingale representation theorem find some $\beta \in L(B)$ such that

$$Z_T := \frac{dQ}{dP} = 1 + \int_0^T \beta_s dB_s.$$

(note that $E[Z_T] = 1$).

Moreover, $\beta \bullet B$ is a martingale and so we have the equality

$$Z_t := E[Z_T \mid \mathcal{F}_t] = 1 + \int_0^t \beta_s dB_s.$$

Note that $Z_t \geq 0$, since the same is true of Z_T. Z is also continuous. Moreover, since Z is a martingale (being a supermartingale suffices for this), we obtain that if $Z_t = 0$ for some $t \geq 0$ then $Z_s = 0$ for all $s \in [t, T]$. This implies that $\beta_s = 0$ for all $s \in [t, T]$.

From these considerations we deduce that we can find $\lambda_s \in L(B)$ such that $\beta_s = \lambda_s Z_s$, and we obtain

$$Z_t = 1 + \int_0^t \lambda_s Z_s dB_s = \mathcal{E}\left(\int_0^\cdot \lambda_s dB_s\right)_t.$$

This yields in particular the result we want.

(b) From Girsanov's theorem, we would expect that $\tilde{B}_t = B_t - \int_0^t \lambda_s^{\text{tr}} ds$ is a Brownian motion under Q. We try to show this directly. Consider some $u \in \mathbb{R}^n$ and some $t \in [0, T]$. Now consider the following:

$$\begin{aligned}
&E_Q\left[\exp\left(i\left(u\cdot B_T^t-\int_0^t u\cdot\lambda_s^{\mathrm{tr}}ds\right)+t\frac{||u||^2}{2}\right)\right]\\
&=E_P\left[\exp\left(\int_0^T\lambda_s dB_s-\frac{1}{2}\int_0^T||\lambda_s||^2ds+i\left(u\cdot B_T^t-\int_0^t u\cdot\lambda_s^{\mathrm{tr}}ds\right)+t\frac{||u||^2}{2}\right)\right]\\
&=E_P\left[\exp\left(M_T-\frac{1}{2}\langle M\rangle_T\right)\right]\\
&=1,
\end{aligned}$$

where $M=\int_0^\cdot \lambda_s dB_s+iu\cdot B^{t_k}$ is a local P-martingale. The last equality holds since $\mathcal{E}(M)$ is a martingale (clearly it is a local martingale, and it is a true martingale by comparison with Z, which we know to be one).

This holds for any u, and by inspecting the first line we conclude that we computed the characteristic function of $\tilde{B}_t$ under Q, and in particular

$$\tilde{B}_t \overset{Q}{\sim} \mathcal{N}(0,tI)$$

where I is the identity matrix.

By a similar computation, we can conclude that the increments of $\tilde{B}$ are independent under Q. Since $\tilde{B}$ is continuous (a.s under P and Q), this shows that $\tilde{B}_t$ is a Brownian motion under Q.