Mathematical Finance

Exercise sheet 8

Exercise 8.1 As in exercise 2 of sheet 6, let $W = W_0 + (W^1, W^2, W^3)$ be a Brownian motion in $\mathbb{R}^3$, i.e. W^1, W^2, W^3 are independent Brownian motions and $W_0 \in \mathbb{R}^3 \setminus \{0\}$ is an $\mathcal{F}_0$-measurable random variable.

Consider the market defined by a riskless asset S^0 with constant price 1, and the risky asset with price process

$$S_t = |W_t|.$$

Show that this market satisfies (NUPBR) but not (NA).

Solution 8.1 As in exercise 2 of sheet 6, we know that $Y_t = S_t^{-1}$ is a strictly positive strict local martingale. Moreover, we have that $S_t Y_t = 1$, so that Y is a local martingale deflator for S. Therefore it follows by the same argument as in question 2(b) of sheet 7 that (NUPBR) is satisfied.

To show that (NA) does not hold, we show that there does not exist an equivalent martingale measure. First, we note by Itô's formula that

$$dS_t = d\tilde{W}_t + \frac{1}{S_t} dt,$$

where $\tilde{W}$ is a Brownian motion (by Lévy's characterisation). In other words, S satisfies the SDE

$$S_t = |W_0| + \int_0^t \left(d\tilde{W}_u + \frac{1}{S_u} du \right).$$

One can show that S is a strong solution, meaning that S is adapted with respect to the augmented filtration generated by W_0 and $\tilde{W}$.

One can also show (by a martingale representation theorem) that any equivalent local martingale measure must be of the form

$$g(W_0) Y_T \mathcal{E}(\theta \bullet W'_T),$$

where g is strictly positive with $E[g(W_0)] = 1$, and W' is a two-dimensional Brownian motion orthogonal to $\tilde{W}$, with $\theta \in L(W')$. However, this is not an equivalent measure, since its density process is only a local martingale. This follows from the fact that

$$\begin{aligned} E\left[g(W_0) Y_T \mathcal{E}(\theta \bullet W'_T)\right] &= E\left[g(W_0) Y_T E[\mathcal{E}(\theta \bullet W'_T) \mid \sigma(W_0, \{\tilde{W}_s, s \leq T\})]\right] \\ &\leq E\left[g(W_0) Y_T\right] \\ &= E\left[g(W_0) E[Y_T \mid \sigma(W_0)]\right] \\ &< E[g(W_0)] = 1, \end{aligned}$$

so that it cannot be a true martingale.

Exercise 8.2 Suppose we define a model with time interval $[0, 1]$, one riskless asset of constant price 1 and one risky asset which is a compound Poisson process with standard normal jumps.

Specifically, for some Poisson process $(N_t)_{t\in[0,1]}$ of rate 1 and $(Z_k)_{k\in\mathbb{N}}$ a sequence of i.i.d. standard normal variables (also independent from N), we have that

$$S_t = \sum_{k=1}^{N_t} Z_k.$$

We take the natural filtration of S.

Show that the only admissible strategy is 0.

Solution 8.2 Let ϑ be an admissible strategy, so that it is predictable and $\vartheta \bullet S \geq -M$ for some $M > 0$ a.s.. Suppose that ϑ is not 0, in the sense that $\vartheta \neq 0$ $dt \times dP$-a.s..

Let $\tau_1, \tau_2, ...$ be the stopping times at which the Poisson process jumps, and take $\tau_0 = 0$. Noting that N has predictable compensator t, we have that

$$E[(|\vartheta| \bullet N)] = E[(|\vartheta| \bullet t)] > 0,$$

by assumption. This means that $E[\mathbb{1}_{\tau_k \leq 1}|\vartheta_{\tau_k}|] > 0$ for some k. In particular we may assume that $P(\tau_k \leq 1, \vartheta_{\tau_k} > \epsilon) > 0$ for some $\epsilon > 0$ (the case $P(\tau_k \leq 1, \vartheta_{\tau_k} < -\epsilon) > 0$ is similar).

Now, note that

$$\begin{aligned} E\left[\mathbb{1}_{\tau_k\leq 1}\mathbb{1}_{\vartheta_{\tau_k}>\epsilon}\mathbb{1}_{Z_k<-\frac{1}{\epsilon}(M+(\vartheta\bullet S)_{\tau_{k-1}})}\right] &= E\left[E\left[\mathbb{1}_{\tau_k\leq 1}\mathbb{1}_{\vartheta_{\tau_k}>\epsilon}\mathbb{1}_{Z_k<-\frac{1}{\epsilon}(M+(\vartheta\bullet S)_{\tau_{k-1}})} \mid \mathcal{F}_{\tau_k-}\right]\right] \\ &= E\left[\mathbb{1}_{\tau_k\leq 1}\mathbb{1}_{\vartheta_{\tau_k}>\epsilon}\Phi\left(-\frac{1}{\epsilon}(M+(\vartheta\bullet S)_{\tau_{k-1}})\right)\right], \end{aligned}$$

where Φ is the cdf of the standard normal distribution. In the above we used that, since ϑ is predictable, ϑ_{τ_k} is $\mathcal{F}_{\tau_k-}$-measurable.

Now, since $P(\mathbb{1}_{\tau_k\leq 1}\mathbb{1}_{\vartheta_{\tau_k}>\epsilon} > 0) > 0$ and $\Phi > 0$, we obtain that the above expectation is positive. We have therefore concluded that

$$P(\tau_k \leq 1, \vartheta_{\tau_k} > \epsilon, Z_k < -\frac{1}{\epsilon}(M + (\vartheta \bullet S)_{\tau_{k-1}})) > 0.$$

But it is clear that this event implies that $(\vartheta \bullet S)_{\tau_k} < -M$. This contradicts the assumption of admissibility.

Exercise 8.3 Consider a discrete time setting with deterministic time points $0 = t_0 < t_1 < t_2 < \dots < t_n = T$. In this setting, semimartingales are given in the form

$$S = \sum_{k=0}^{n-1} S_k \mathbb{1}_{[t_k, t_{k+1})} + S_n \mathbb{1}_{\{T\}},$$

where each S_k is $\mathcal{F}_{t_k}$-measurable.

Show that in this case, ucp convergence is equivalent to convergence in Emery topology.

Solution 8.3 It is clear that convergence in Emery topology implies ucp convergence, since the Emery metric is stronger. We can see this by the inequality

$$d_E(S, 0) = \sup_{H \in \mathcal{S}_1} E[1 \wedge (H \bullet S)^*_T] \geq E[1 \wedge S^*_T] = d(S, 0),$$

where d_E and d are the metrics for Emery topology and ucp topology (up to equivalence).

For the converse, note that

$$\begin{aligned}
d_E(S, 0) &= \sup_{H \in \mathcal{S}_1} E[1 \wedge (H \bullet S)^*_T] \\
&= \sup_{H \in \mathcal{S}_1} E\left[1 \wedge \sup_{k \in \{0, \dots, n-1\}} \left| \sum_{i=0}^{k} H_i (S_{i+1} - S_i) \right| \right] \\
&\leq \sup_{H \in \mathcal{S}_1} E\left[1 \wedge \sup_{k \in \{0, \dots, n-1\}} \sum_{i=0}^{k} |H_i| |S_{i+1} - S_i| \right] \\
&\leq 2n E[1 \wedge S^*_T] \\
&= 2n\, d(S, 0).
\end{aligned}$$

Here the n is fixed, therefore we also have that d_E is weaker than d (in this setup), which gives equivalence.

Exercise 8.4 Show that

(a) A local martingale is a sigma-martingale.

(b) A sigma-martingale which is also a special semimartingale is a local martingale.

Solution 8.4

(a) It is sufficient to show that the requirement that the martingale M in the sigma-martingale representation $X = H \bullet M$ with a predictable $H > 0$ can be relaxed to M a local martingale. Let $\tau_0 = 0$ and $(\tau_n)_{n\in\mathbb{N}}$ be the localizing sequence for M in $\mathcal{H}^1$. For each n, set $N^n := \mathbb{1}_{(\tau_{n-1},\tau_n]} \bullet M^{\tau^n}$ and choose $\alpha_n > 0$ such that $\sum_n \alpha_n ||N^n||_{\mathcal{H}^1} < \infty$. Then $N := \sum_n \alpha_n N^n$ is an $\mathcal{H}^1$-martingale and, for $J := \mathbb{1}_{\{0\}} + H \sum_n \alpha_n^{-1} \mathbb{1}_{(\tau_{n-1},\tau_n]} > 0$, we have $X = J \bullet N$.

(b) Let $X = M + A$, where M is a local martingale and A is a predictable FV process with $A_0 = 0$. It is sufficient to show that $A = 0$. There exists predictable $H > 0$ such that $H \bullet X$ is a (local) martingale and without loss of generality we may assume that H is bounded. Indeed, if $X = \tilde{H} \bullet \tilde{M}$ is the sigma-martingale decomposition and $H := \tilde{H}^{-1}$, we have

$$H \bullet X = \tilde{H}^{-1} \bullet (\tilde{H} \bullet M) = \tilde{M}$$

and

$$(H \wedge 1) \bullet X = \left(\frac{H \wedge 1}{H}\right) \bullet (H \bullet X)$$

which is a local martingale, since $H \bullet X$ is. Note that

$$H \bullet A = (H \bullet A)_- + \Delta(H \bullet A) = (H \bullet A)_- + (H \bullet \Delta A),$$

so the process $H \bullet A$ is predictable. Consequently, the process $H \bullet A = H \bullet X - H \bullet M$ is a predictable FV local martingale, so $H \bullet A = 0$. Since A is a FV process, we can decompose $[0,T]$ into two random sets P and N such that $P \cup N = [0,T]$ and $P \cap N = \emptyset$, such that dA is a (non-negative) measure on P and $-dA$ is a (non-negative) measure on N. Taking $J = \mathbb{1}_P - \mathbb{1}_N$ we get

$$\begin{aligned} 0 &= J \bullet (H \bullet A) \\ &= H \bullet (J \bullet A) \\ &= \int_0^{\cdot} H(\mathbb{1}_P dA - \mathbb{1}_N dA) \\ &\geq 0 \end{aligned}$$

as $H > 0$, with equality in the last line if and only if $dA = 0$. Since $A_0 = 0$, this shows the result.

Exercise 8.5 In the same setup of question 1, consider the Bachelier model:

$$S_t = S_0 + \mu t + \sigma B_t$$

on $[0, T]$, where B is a d-dimensional Brownian motion, $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d\times d}$ is invertible.

(a) Show that there exists a unique equivalent measure Q such that for all $f \in L^\infty(\mathcal{F}_T)$, $E_Q(f) = \pi(f)$, where π is the superreplication price.

(b) Take $d = 1$ and $f = (S_T - K)^+$, for some $K \in \mathbb{R}$. Compute $\pi(f)$ as well as the unique strategy ϑ such that

$$\pi(f) + (\vartheta \bullet S)_T = f.$$

(c) Have a look at this paper and write a very short summary of some of the main points.

Solution 8.5

(a) As in question 4 of sheet 5, any equivalent measure Q can be written in the form

$$\frac{dQ}{dP} = \exp\left(\int_0^T -\lambda_s dB_s - \frac{1}{2}\int_0^T ||\lambda_s||^2 ds\right),$$

and then we have that

$$\tilde{B}_t = B_t + \int_0^t \lambda_s ds$$

is a Q-Brownian motion. It is then clear that

$$\frac{dQ}{dP} = \exp\left(\int_0^T -(\sigma^{-1}\mu)^{\mathrm{tr}} dB_s - \frac{T}{2}||\sigma^{-1}\mu||^2\right)$$

is an equivalent martingale measure since $\tilde{B}_t = B_t + \sigma^{-1}\mu t$ is a Brownian motion under Q.

Conversely, Q is the unique such measure, since under any other equivalent measure we have that $\tilde{B}$ is a Brownian motion with (non-trivial) drift.

Now, since $\tilde{B}$ is a Brownian motion under Q, and $f \in L^\infty(\mathcal{F}_T, Q)$ (since L^∞ is preserved by an equivalent change of measure), we can use the martingale representation theorem to find some θ such that

$$f = E_Q(f) + \int_0^T \theta_s d\tilde{B}_s = E_Q(f) + \int_0^T \vartheta_s dS_s,$$

where $\vartheta_s = \sigma^{-1}\theta_s$. This means that $E_Q(f) \geq \pi(f)$. Conversely, if we have any representation

$$f \leq f_0 + \int_0^T \tilde{\vartheta}_s dS_s = f_0 + \int_0^T \tilde{\theta}_s d\tilde{B}_s,$$

with $\tilde{\theta}_s = \tilde{\vartheta}_s\sigma$, we obtain that $f_0 \geq E_Q(f)$, as $\tilde{\theta} \bullet \tilde{B}$ is a local Q-martingale. Thus $E_Q(f) = \pi(f)$.

Now, consider some other martingale measure $\hat{Q}$ with associated process $\hat{\lambda}$, and let $\tilde{\lambda} := \sigma^{-1}\mu$. Take

$$f_n = ((\tilde{\lambda} - \hat{\lambda})^\top \sigma^{-1} \bullet S)_{\tau_n},$$

where $\tau_n := \inf\{t \geq 0 : |((\tilde{\lambda} - \hat{\lambda})^\top \sigma^{-1} \bullet S)_t| \geq n\} \wedge T$. By continuity and choice of τ_n, f_n is bounded. Note that

$$\begin{aligned}
f_n &= \int_0^T \mathbb{1}_{[0,\tau_n]}(\tilde{\lambda} - \hat{\lambda}_s)^\top \sigma^{-1} dS_s \\
&= \int_0^T \mathbb{1}_{[0,\tau_n]}(\tilde{\lambda} - \hat{\lambda}_s)^\top d\tilde{B}_s \\
&= \int_0^T \mathbb{1}_{[0,\tau_n]}(\tilde{\lambda} - \hat{\lambda}_s)^\top dB_s + \int_0^T \mathbb{1}_{[0,\tau_n]}(\tilde{\lambda} - \hat{\lambda}_s)^\top \sigma^{-1}\mu ds \\
&= \int_0^T \mathbb{1}_{[0,\tau_n]}(\tilde{\lambda} - \hat{\lambda}_s)^\top d\hat{B}_s + \int_0^T \mathbb{1}_{[0,\tau_n]}(\tilde{\lambda} - \hat{\lambda}_s)^\top (\sigma^{-1}\mu - \hat{\lambda}_s) ds \\
&= \int_0^T \mathbb{1}_{[0,\tau_n]}(\tilde{\lambda} - \hat{\lambda}_s)^\top d\hat{B}_s + \int_0^T \mathbb{1}_{[0,\tau_n]}(\tilde{\lambda} - \hat{\lambda}_s)^\top (\tilde{\lambda} - \hat{\lambda}_s) ds.
\end{aligned}$$

But then, the finite variation term is increasing. Up to localisation, we can take the first term to be a $\hat{Q}$-martingale, and so $E_{\hat{Q}}(f_n) \geq 0$, strictly unless the finite variation term is 0. In order to have equality for all f, and in particular for all f_n, we obtain that $\hat{\lambda}_s = \tilde{\lambda} = -(\sigma^{-1}\mu)^{\mathrm{tr}}$, which gives uniqueness of Q satisfying the desired properties.

(b) Working under Q, we first want to compute

$$\pi(f) = E_Q[(S_0 + \sigma \tilde{B}_T - K)^+].$$

This is the same as

$$\pi(f) = \sigma\sqrt{T} E_Q \left[\left(\frac{\tilde{B}_T}{\sqrt{T}} - \frac{(K - S_0)}{\sigma\sqrt{T}} \right)^+ \right].$$

Letting

$$\psi(x) = \int_x^\infty \frac{(y-x)}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy,$$

we obtain that

$$\pi(f) = \sigma\sqrt{T}\psi\left(\frac{K - S_0}{\sigma\sqrt{T}}\right).$$

More generally, we obtain

$$f_t := E_Q[f \mid \mathcal{F}_t] = \sigma\sqrt{T-t}\,\psi\left(\frac{K - S_0 - \sigma\tilde{B}_t}{\sigma\sqrt{T-t}}\right),$$

for $t < T$. We can then use Itô's formula, as well as the fact that $\psi'(x) = \Phi(x) - 1$ for Φ the cdf of the normal distribution, to obtain

$$df_t = \left(1 - \Phi\left(\frac{K - S_0 - \sigma\tilde{B}_t}{\sigma\sqrt{T-t}}\right)\right)\sigma d\tilde{B}_t,$$

and therefore we obtain the representation

$$f = \pi(f) + \int_0^T \vartheta_s dS_s$$

where

$$\vartheta_t = \left(1 - \Phi\left(\frac{K - S_t}{\sigma\sqrt{T-t}}\right)\right).$$

Exercise 8.6 (Python) Assume Black-Scholes dynamics for S, say $(r, \mu, \sigma) = (0, 0, 1)$, and find the hedging strategy H for the log-contract g whose discounted payoff is given by

$$g(S_T) = \log \frac{S_T}{S_0} + \frac{1}{2}\sigma^2 T.$$

Compare numerically the value of $g(S_T)$ to $(H \bullet S)_T$ at $T = 1$.

References

[1] Walter Schachermayer; Josef Teichmann. *How close are the option pricing formulas of Bachelier and Black-Merton-Scholes?* Mathematical Finance, 18: 155-170. doi:10.1111/j.1467-9965.2007.00326.x