Mathematical Finance

Exercise sheet 9

Exercise 9.1

(a) Let $\mathcal{C} \subseteq L^0$ be non-empty, closed, convex and bounded. Suppose that $J : L^0 \to \mathbb{R}$ is a continuous strictly concave function such that

$$\sup_{g \in \mathcal{C}} J(g) < \infty,$$

and that $J \not\equiv -\infty$ on $\mathcal{C}$. Show that J has a unique maximiser $\hat{g}$.

(b) Let $(\mathcal{C}_n)_{n\in\mathbb{Z}}$ be an increasing sequence of closed, convex, bounded subsets of L^0, i.e. such that $\mathcal{C}_n \subseteq \mathcal{C}_m$ for $n \leq m$. Show that J has a unique maximiser $\hat{g}_{-\infty}$ on

$$\mathcal{C}_{-\infty} := \bigcap_{n\in\mathbb{Z}} \mathcal{C}_n$$

if $\mathcal{C}_{-\infty}$ is non-empty and $J \not\equiv -\infty$ on $\mathcal{C}_{-\infty}$, and moreover there exists a sequence $(\tilde{h}_n)$ of forward convex combinations of $(\hat{g}_{-n})_{n\geq 1}$ such that $\tilde{h}_n \to \hat{g}_{-\infty}$ as $n \to \infty$.

(c) Suppose that $\bigcup_{n\in\mathbb{Z}} \mathcal{C}_n$ is bounded in probability, and that f is bounded above in this set. Show that J has a unique maximiser $\hat{g}_\infty$ on

$$\mathcal{C}_\infty = \overline{\bigcup_{n\in\mathbb{Z}} \mathcal{C}_n}^{L^0},$$

and moreover $\hat{g}_n \to \hat{g}_\infty$ as $n \to \infty$.

(d) Suppose that J is uniformly strictly concave, in the sense that there exists a continuous strictly increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(0) = 0$ and such that for $f_1, f_2 \in L^0$,

$$J\left(\frac{f_1 + f_2}{2}\right) - \frac{J(f_1) + J(f_2)}{2} \geq \rho(d(f_1, f_2)).$$

Show that in (b) and (c), we already have that $\hat{g}_{-n} \to g_{-\infty}$ and $\hat{g}_n \to g_\infty$, respectively.

Solution 9.1

(a) Let g^n be a maximising sequence, i.e. such that $J(g^n) \uparrow \alpha := \sup_{g\in\mathcal{C}} J(g) < \infty$. We also have $\alpha > -\infty$. By the Komlós lemma, since all the g^n are contained in $\mathcal{C}$ which is bounded in probability, we can find a sequence of forward convex combinations $\tilde{g}^n$ such that $\tilde{g}^n \to g$ in probability for some $g \in L^0$.

Since $\mathcal{C}$ is convex, each $\tilde{g}^n \in \mathcal{C}$, and since $\mathcal{C}$ is closed we obtain that $g \in \mathcal{C}$. Moreover, since J is concave and $(J(g^n))$ is increasing, we have that $J(\tilde{g}^n) \geq J(g^n)$ for each n, so that $J(\tilde{g}^n) \uparrow \alpha$. Finally, by continuity, we obtain that $J(g) = \alpha$.

To show uniqueness, we observe that if g and g' are two maximisers, then $J\left(\frac{g+g'}{2}\right) > \frac{1}{2}(J(g) + J(g')) = \alpha$ by strict concavity, which contradicts the optimality.

(b) It is clear that $\mathcal{C}_{-\infty}$ is convex (as an intersection of convex sets), closed (as an intersection of closed sets) and bounded (since it is contained in some $\mathcal{C}^n$). Likewise, J is bounded above and not identical to $-\infty$ on $\mathcal{C}_{-\infty}$, therefore by (a) it follows that there exists a unique maximiser $g_{-\infty}$.

From the sequence $h_n := g_{-n}$ (for $n \geq 1$) we can find a sequence of forward convex combinations $\tilde{h}_n$ such that $\tilde{h}_n \to g \in L^0$. By convexity and closedness, we have that each $\mathcal{C}_{-m}$ contains a tail of $(\tilde{h}_n)$, and therefore $g \in \mathcal{C}_{-m}$. Since this holds for each m, we deduce that $g \in \mathcal{C}_{-\infty}$.

Next, we show that $g = \hat{g}_{-\infty}$. Indeed, by concavity we have that

$$J(\tilde{h}_n) \geq J(\hat{g}_{-l_n}) \geq J(\hat{g}_{-\infty}),$$

since $\mathcal{C}_{-m_n} \supseteq \mathcal{C}_{-\infty}$ and where $\tilde{h}_n$ is a convex combination of $h_n, ..., h_{l_n}$. By continuity, this yields that $J(g) \geq J(\hat{g}_{-\infty})$ and by uniqueness of the optimiser, we get that $g = g_{-\infty}$.

(c) We start by showing the existence of a unique maximiser. Note that $\mathcal{C}_\infty$ is closed, by assumption, and we have that $\bigcup_{n\in\mathbb{Z}} \mathcal{C}_n$ is convex and bounded in probability. Convexity of C_∞ follows easily, since if $f = \sum_{j=1}^k \lambda^j f^j$ with $f^j \in C_\infty$, $\lambda^j \geq 0$ and $\sum_j \lambda^j = 1$, then $f = \lim_{n\to\infty} \sum_{j=1}^k \lambda^j f_n^j \in \mathcal{C}_\infty$ where $f_n^j \to f^j$ are taken in $\bigcup_{n\in\mathbb{Z}} \mathcal{C}_n$.

Likewise, boundedness in probability follows since for $f \in \mathcal{C}_\infty$,

$$P(|f| > K+1) \leq P(|f'| > K) + P(|f - f'| > 1 \leq 2\epsilon$$

for K large enough and $f' \in \bigcup_{n\in\mathbb{Z}} \mathcal{C}_n$ close enough to f.

Therefore, $\mathcal{C}_\infty$ satisfies the required conditions for (a), and there exists a unique maximiser $\hat{g}_\infty$.

Next, we show that $J(\hat{g}_n) \uparrow J(\hat{g}_\infty)$ as $n \to \infty$. This follows from the following observation:

$$\begin{aligned}
\lim_{n\to\infty} J(\hat{g}_n) &= \sup_{n\in\mathbb{Z}} J(\hat{g}_n) \\
&= \sup_{f\in\bigcup_{n\in\mathbb{Z}} \mathcal{C}_n} J(f) \\
&= \sup_{f\in\mathcal{C}_\infty} J(f), \\
&= J(\hat{g}_\infty)
\end{aligned}$$

using continuity of J for the next to last step.

Since $\mathcal{C}_\infty$ is bounded in probability, we can find a sequence of forward convex combinations $\tilde{g}_n$ such that $\tilde{g}_n \to g \in C_\infty$. We just need to show that $g = \hat{g}_\infty$. That follows from the fact that $J(\tilde{g}_n) \geq J(\tilde{g}_n) \uparrow J(\hat{g}_\infty)$ (by concavity), so that $J(g) \geq J(\hat{g}_\infty)$ by continuity and therefore $g = \hat{g}_\infty$, by uniqueness of the maximiser.

(d) For (b), we show that $(\hat{g}_{-n})$ is a Cauchy sequence in L^0, so that it has a limit and by the same argument as in (b), the limit must be equal to $\hat{g}_{-\infty}$.

Note that, using (b) we can show that

$$\alpha_{-n} = J(\hat{g}_{-n}) \downarrow J(\hat{g}_{-\infty}) = \alpha_{-\infty}.$$

Now, for any $n \geq m$,

$$0 \leq J(\hat{g}_{-m}) - J(\hat{g}_{-n}) \leq \alpha_{-m} - \alpha_{-\infty} \downarrow 0.$$

Let m be large enough that $\alpha_{-m} - \alpha_{-\infty} < \epsilon$, for some given small $\epsilon > 0$. Noting that $\hat{g}_{-n} \in \mathcal{C}_{-m}$, we also obtain that $\frac{\hat{g}_{-n}+\hat{g}_{-m}}{2} \in \mathcal{C}_{-m}$, and therefore by optimality of $\hat{g}_{-m}$ and the uniform strict concavity assumption,

$$\begin{aligned} J(\hat{g}_{-m}) &\geq J\left(\frac{\hat{g}_{-n}+\hat{g}_{-m}}{2}\right) \\ &\geq \frac{J(\hat{g}_{-n})+J(\hat{g}_{-m})}{2} + \rho(d(\hat{g}_{-n}, \hat{g}_{-m})). \end{aligned}$$

Rearranging,

$$\epsilon > 2\rho(d(\hat{g}_{-n}, \hat{g}_{-m})).$$

Thanks to the hypotheses on ρ, this shows that $(\hat{g}_{-n})$ is a Cauchy sequence in L^0, which by the above arguments is enough to conclude. The argument for (c) is very similar.

Exercise 9.2

(a) Consider the Bachelier model

$$S_t = S_0 + \sigma B_t$$

in its natural filtration on the interval $[0, T]$. Prove the martingale representation theorem for bounded martingales, using the fact that the set of equivalent separating measures is a singleton.

(b) Consider a continuous trajectory model S for a d-dimensional discounted price process in its natural filtration. Assume that there is only one equivalent separating measure. Prove that martingale representation holds true for bounded martingales.

(c) Consider a finite filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supporting a model S for a d-dimensional discounted price process. Assume that there exists more than one equivalent separating measure. Prove that there is at least one bounded claim g such that either g or $-g$ cannot be replicated.

Solution 9.2

(a) We note that there exists only one equivalent local martingale measure, namely P itself. If Q is an equivalent local martingale measure, then we have that B is a local martingale under Q and its quadratic variation is $\langle B \rangle_t = t$, since it is preserved under equivalent changes of measure. Therefore, B is a Brownian motion under Q. Since the law of B determines a unique probability measure on its natural filtration, it follows that $Q = P$.

Suppose now that Q is an equivalent separating measure. For $\tau_n = \inf\{t \geq 0 : |B_t| \geq n\}$, one easily obtains that B^{τ_n} must be a martingale under Q for each n (by the separating property and considering both B^{τ_n} and $-B^{\tau_n}$). Therefore, Q is a local martingale measure for W, and $Q = P$ by the above.

Finally, if Q is an absolutely continuous separating measure, then $\lambda Q + (1 - \lambda)P$ is an equivalent separating measure for each $\lambda \in [0, 1]$, whence $Q = P$.

Next, let ξ be a bounded $\mathcal{F}_T$-measurable random variable. Let $c = E_P[\xi]$. Consider the set

$$C := \{\theta \bullet S_T : \theta \bullet S \text{ is bounded}\} - L^\infty_+ \subseteq L^\infty.$$

This set C is a weak-$*$-closed convex cone. Its double dual is therefore $C^{00} = C$ while its dual (in L^1) is the cone generated by the densities of absolutely continuous separating measures, so that $C^0 = \mathbb{R}_+ \mathbb{1}$.

We want to show that $\xi - c \in C$. Indeed, it is enough to show that $\xi - c \in C^{00}$, and this follows from the fact that $E_P[\xi - c] = 0$. Likewise, $-\xi + c \in C$, since $E_P[-\xi + c] = 0$. Therefore, we can find $\vartheta_1, \vartheta_2 \in L(S)$ such that $\vartheta_1 \bullet S, \vartheta_2 \bullet S$ are bounded and nonnegative bounded random variables g_1, g_2 such that

$$\xi - c = \vartheta_1 \bullet S_T - g_1$$

and

$$-\xi + c = \vartheta - 2 \bullet S_T - g_2.$$

But then,

$$(\vartheta_1 + \vartheta_2) \bullet S_T = g_1 + g_2 \geq 0$$

so that $g_1 + g_2 = 0$ and $(\vartheta_1 + \vartheta_2) \bullet S_T = 0$, by no arbitrage. We conclude in particular that $\xi = c + \vartheta_1 \bullet S_T$ can be replicated.

In the case of M being a bounded martingale, one can use the above to replicate $M_T = c + \vartheta \bullet S_T$ where $\vartheta \bullet S$ is bounded. By taking conditional expectations, we conclude that $M = c + \vartheta \bullet S$.

(b) The argument is very similar. Once again, uniqueness of an absolutely continuous separating measure follows by taking convex combinations with the equivalent separating measure. By the same reasoning, we can show that $\xi - E_P[\xi] \in C$ and therefore any bounded payoff (and any bounded martingale) can be replicated.

(c) Let P and Q be two distinct equivalent local martingale measures. Then, there is some $\omega \in \Omega$ such that $P(\omega) > Q(\omega)$. We claim that $\mathbb{1}_\omega$ cannot be replicated. Indeed, if it can, there exists some $c \in \mathbb{R}$ and strategy ϑ such that $\vartheta \bullet S_T = \mathbb{1}_\omega$, and $\vartheta \bullet S$ is bounded. Now, since P and Q are separating measures, we obtain

$$P(\omega) = c + E_P[\vartheta \bullet S_T] \leq c$$

and

$$-Q(\omega) = -c + E_Q[-\vartheta \bullet S_T] \leq -c$$

so that

$$P(\omega) \leq c \leq Q(\omega),$$

which gives a contradiction.

Exercise 9.3 Consider a general model, with $[0, 1]$ as the time interval, a riskless asset of constant price 1, and some d-dimensional semimartingale S representing the prices of the risky assets.

Define

$$G = \{(\vartheta \bullet S)_T, \vartheta \in \Theta_{adm}\} \subseteq L^0$$

and

$$C = (G - L^0_{\geq 0}) \cap L^\infty \subseteq L^\infty.$$

(a) Show that the following notions of no arbitrage are equivalent:

$$G \cap L^0_{\geq 0} = \{0\}$$

and

$$C \cap L^\infty_{\geq 0} = \{0\}.$$

(b) Prove that C is weak-$*$-closed if and only for any bounded sequence (f_n) in C converging almost surely to f_0, it holds that $f_0 \in C$.

Solution 9.3

(a) $\Rightarrow$: Suppose $x \in C \cap L^\infty_{\geq 0}$. By definition of C, there exists some $y \in G$ with $y \geq x \geq 0$. But then, by assumption, $y = 0$ so that $x = 0$.

$\Leftarrow$: Suppose $y \in G \cap L^0_{\geq 0}$. Then let $x = y - (y - 1)^+$. Note that $x \in C$ since $(y - 1)^+ \in L^0_{\geq 0}$, and $x \in L^\infty$ as $x \leq 1$. If $y \neq 0$ then $x \neq 0$, which contradicts the assumption.

(b) $\Rightarrow$: Suppose x_n are in C and $x_n \to x$ almost surely. Since the x_n are bounded by some $M > 0$, for any $Z \in L^1$, we have that each $|Zx_n| \leq M|Z|$ and so, by DCT, $E[Zx_n] \to E[Zx]$. Therefore $x_n \to x$ in weak-$*$-topology and so $x \in C$.

$\Leftarrow$: Suppose x_n are in C and $x_n \to x$ in weak-$*$ topology. Weak-$*$-convergence implies that the sequence is bounded in L^∞, say by M, and of course also in L^0. By Komlos' lemma, we can find a sequence of forward convex combinations y_n in C with $y_n \to y$ almost surely; moreover the y_n are bounded by some $M > 0$ since the x_n are. But then, for any $Z \in L^1$, we have that each $|Zy_n| \leq M|Z|$ and so, by DCT, $E[Zy_n] \to E[Zy]$. This means that $y_n \to y$ in weak-$*$-topology and so $y = x$. Since y is the almost sure limit of the y_n, which belong to C, then $x = y \in C$ by assumption.