Multilinear Algebra and Its Applications

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There are two alternative ways of finding L:

(1) *With matrix inversion:* Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \tag{1.7}$$

where $D = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the *determinant*. Thus

$$L = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(2) *With Gauss-Jordan elimination:*

$$\left[\begin{array}{cc|cc} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \end{array}\right] \quad \leadsto \quad \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

□

1.3.3. The Kronecker Delta Symbol.

NOTATION. The **Kronecker delta symbol** δ^i_j is defined as

$$\delta^i_j := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,. \end{cases} \tag{1.8}$$

EXAMPLES 1.26. If L is a matrix, the (i,j)-entry of L is the coefficient in the i-th row and j-th column, and is denoted by L^i_j.

(1) The $n \times n$ **identity matrix**

$$I = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}$$

has (i,j)-entry equal to δ^i_j.

(2) Let L and M be two square matrices. The (i,j)-th entry of the product

$$ML = \begin{bmatrix} M^1_1 & \dots\dots & M^1_n \\ \vdots & & \vdots \\ M^i_1 & \dots\dots & M^i_n \\ \vdots & & \vdots \\ M^n_1 & \dots\dots & M^n_n \end{bmatrix} \begin{bmatrix} L^1_1 \dots & L^1_j & \dots L^1_n \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ L^n_1 \dots & L^n_j & \dots L^n_n \end{bmatrix}$$

equals the *dot product* of the i-th $\overbrace{\text{row}}^{\text{of } M}$ and j-th $\overbrace{\text{column}}^{\text{of } L}$,

$$\begin{pmatrix} M^i_1 & \dots & M^i_n \end{pmatrix} \cdot \begin{pmatrix} L^1_j \\ \vdots \\ L^n_j \end{pmatrix} = M^i_1 L^1_j + \dots + M^i_n L^n_j\,,$$

CHAPTER 2

Multilinear Forms

2.1. Linear Forms

2.1.1. Definition and Examples.

Definition 2.1. Let V be a vector space. A **linear form** on V is a map $\alpha : V \to \mathbb{R}$ such that for every $a, b \in \mathbb{R}$ and for every $v, w \in V$

$$\alpha(av + bw) = a\alpha(v) + b\alpha(w)\,.$$

Alternative terminologies for "linear form" are **tensor of type** $(0,1)$, **1-form**, **linear functional** and **covector**.

Exercise 2.2. If $V = \mathbb{R}^3$, which of the following are linear forms?

(1) $\alpha(x,y,z) := xy + z$;
(2) $\alpha(x,y,z) := x + y + z + 1$;
(3) $\alpha(x,y,z) := \pi x - \frac{7}{2}z$.

Exercise 2.3. If V is the infinite dimensional vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$, which of the following are linear forms?

(1) $\alpha(f) := f(7) - f(0)$;
(2) $\alpha(f) := \int_0^{33} e^x f(x) dx$;
(3) $\alpha(f) := \cancel{e^{f(x)}}$. $e^{f(4)}$

Example 2.4. [Coordinate forms] This is a most important example of linear form. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of V and let $v = v^i b_i \in V$ be a generic vector. Define $\beta^i : V \to \mathbb{R}$ by

$$\beta^i(v) := v^i\,, \tag{2.1}$$

that is β^i will extract the i-th coordinate of a vector with respect to the basis $\mathcal{B}$. The linear form β^i is called **coordinate form**. Notice that

$$\beta^i(b_j) = \delta^i_j\,, \tag{2.2}$$

since the i-th coordinate of the basis vector b_j with respect to the basis $\mathcal{B}$ is equal to 1 if $i = j$ and 0 otherwise. □

Remark 2.8. Just like any function, two linear forms on V are equal if and only if their values are the same when applied to *each* vector in V. However, because of the defining properties of linear forms, to determine whether two linear forms are equal, it is *enough to check that they are equal on the elements of the basis of* V. In fact, let $\alpha, \alpha' \in V^*$, let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis of V and let us suppose that we know that

$$\alpha(b_j) = \alpha'(b_j)$$

for all $1 \leq j \leq n$. We verify that this implies that they are the same when applied to each vector $v \in V$. In fact let $v = v^j b_j$ its representation with respect to the basis $\mathcal{B}$. Then we have

$$\alpha(v) = \alpha(v^j b_j) = v^j \alpha(b_j) = v^j \alpha'(b_j) = \alpha'(v^j b_j) = \alpha'(v)\,.$$

□

Proposition 2.9. *Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis of V and $\beta^1, \dots, \beta^n$ the corresponding coordinate forms. Then $\mathcal{B}^* := \{\beta^1, \dots, \beta^n\}$ is a basis of V^*. As a consequence*

$$\dim V = \dim V^*\,.$$

Proof. According to Definition 1.17, we need to check that the linear forms in $\mathcal{B}^*$

(1) are linearly independent and
(2) span V^*.

(1) We need to check that the only linear combination of $\beta^1, \dots, \beta^n$ that yields the zero linear form is the trivial linear combination. Let $c_i \beta^i = 0$ be a linear combination of the β^i. Then for every basis vector b_j, with $j = 1, \dots, n$,

$$0 = (c_i \beta^i)(b_j) = c_i(\beta^i(b_j)) = c_i \delta^i_j = c_j\,,$$

thus showing the linear independence.

(2) To check that $\mathcal{B}^*$ spans V we need to verify that any $\alpha \in V^*$ is a linear combination of $\beta^1, \dots, \beta^n$, that is that we can find $\alpha_i \in \mathbb{R}$ such that

$$\alpha = \alpha_i \beta^i \tag{2.3}$$

To find such α_i we apply both sides of (2.3) to the j-th basis vector b_i, and we obtain

$$\alpha(b_j) = \alpha_i \beta^i(b_j) = \alpha_i \delta^i_j = \alpha_j\,, \tag{2.4}$$

which identifies the coefficients in (2.3).

By hypothesis α is a linear form and, since V^* is a vector space, also $\alpha(b_i)\beta^i$ is a linear form. Moreover, we have just verified that these two linear form coincide on the basis vectors. By Remark 2.8 the two linear forms are the same and, hence, we have written α as a linear combination of the coordinate forms. This completes the proof that the coordinate forms form a basis of the dual. □

The basis $\mathcal{B}^*$ of V^* is called the **basis of V dual to $\mathcal{B}$**. We emphasize that the coordinates (or components) of a linear form α with respect to $\mathcal{B}^*$ are exactly the values of α on the elements of $\mathcal{B}$,

$$\alpha_i = \alpha(b_i)\,.$$

EXAMPLE 2.10. Let $V = \mathbb{R}[x]_2$ be the vector space of polynomials of degree ≤ 2, let $\alpha : V \to \mathbb{R}$ be the linear form given by

$$\alpha(p) := p(2) - p'(2) \tag{2.5}$$

and let $\mathcal{B}$ be the basis $\{1, x, x^2\}$ of V. In this example, we want to:

(1) find the components of α with respect to $\mathcal{B}^*$;
(2) describe the basis $\mathcal{B}^* = \{\beta^1, \beta^2, \beta^3\}$;

(1) Since

$$\begin{aligned} \alpha_1 &= \alpha(b_1) = \alpha(1) = 1 - 0 = 1 \\ \alpha_2 &= \alpha(b_2) = \alpha(x) = 2 - 1 = 1 \\ \alpha_3 &= \alpha(b_3) = \alpha(x^2) = 4 - 4 = 0\,, \end{aligned}$$

then

$$[\alpha]_{\mathcal{B}^*} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\,. \tag{2.6}$$

(2) The generic element $p(x) \in \mathbb{R}[x]_2$ written as combination of basis elements $1, x$ and x^2 is

$$p(x) = a + bx + cx^2\,.$$

Hence $\mathcal{B}^* = \{\beta^1, \beta^2, \beta^3\}$, is given by

$$\begin{aligned} \beta^1(a + bx + cx^2) &= a \\ \beta^2(a + bx + cx^2) &= b \\ \beta^3(a + bx + cx^2) &= c\,. \end{aligned} \tag{2.7}$$

□

REMARK 2.11. Note that we have to be careful when referring to a "dual basis" of V^*, as for every basis $\mathcal{B}$ of V there is going to be a basis $\mathcal{B}^*$ of V^* dual to the basis $\mathcal{B}$. In the next section we are going to see how a dual basis transforms with a change of basis.

2.1.3. Covariance of Linear Forms.

We want to examine how a linear form $\alpha : V \to \mathbb{R}$ behaves with respect to a change a basis in V. To this purpose, let

$$\mathcal{B} = \{b_1, \ldots, b_n\} \quad \text{and} \quad \widetilde{\mathcal{B}} := \{\tilde{b}_1, \ldots, \tilde{b}_n\}$$

Therefore, we have

$$\Lambda = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} .$$

(6) The linear form α is *covariant* since

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix} L = \begin{pmatrix}1 & 1 & 0\end{pmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix}1 & 0 & -1\end{pmatrix} = \begin{pmatrix}\widetilde{\alpha}_1 & \widetilde{\alpha}_2 & \widetilde{\alpha}_3\end{pmatrix} .$$

(7) The dual basis $\mathcal{B}^*$ is *contravariant* since

$$\begin{pmatrix}\tilde\beta^1 \\ \tilde\beta^2 \\ \tilde\beta^3\end{pmatrix} = \Lambda \begin{pmatrix}\beta^1 \\ \beta^2 \\ \beta^3\end{pmatrix} ,$$

as it can be verified by ~~looking~~ evaluating at an arbitrary vector $p(x) = a + bx + cx^2$

$$\Lambda \begin{pmatrix}\beta^1(p) \\ \beta^2(p) \\ \beta^3(p)\end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix}a \\ b \\ c\end{pmatrix} = \begin{pmatrix}a+b+c \\ -a-c \\ c\end{pmatrix} = \begin{pmatrix}\tilde\beta^1(p) \\ \tilde\beta^2(p) \\ \tilde\beta^3(p)\end{pmatrix} .$$

□

2.1.4. Contravariance of Dual Bases.

In fact, statement (7) in Example 2.10 holds in general, namely:

PROPOSITION 2.14. *Dual bases are* **contravariant**.

PROOF. We will check that when bases $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are related by

$$\tilde b_j = L^i_j b_i$$

the corresponding dual bases $\mathcal{B}^*$ and $\widetilde{\mathcal{B}}^*$ of V^* are related by

$$\boxed{\tilde\beta^j = \Lambda^j_i \beta^i} . \tag{2.11}$$

It is enough to check that the $\Lambda^j_i\beta^i$ are *dual* to the $\tilde b_j$. In fact, since $\Lambda L = I$, then

$$(\Lambda^k_\ell\beta^\ell)(\tilde b_j) = (\Lambda^k_\ell\beta^\ell)(L^i_j b_i) = \Lambda^k_\ell L^i_j \beta^\ell(b_i) = \Lambda^k_\ell L^i_j \delta^\ell_i = \Lambda^k_i L^i_j = \delta^k_j = \beta^j(\tilde b_j) .$$

□

In Table 1, you can find a summary of the properties that bases and dual bases, coordinate vectors and components of linear forms satisfy with respect to a change of basis and hence whether they are covariant or contravariant. Moreover, Table 2 summarizes the characteristics of covariance and contravariance.

where

$$\sigma = (\sigma(1), \sigma(2), \sigma(3)) \in S_3 := \{\text{permutations of } 3 \text{ elements}\}$$
$$= \{(1,2,3),(1,3,2),(2,3,1),(2,1,3),(3,1,2),(3,2,1)\},$$

and the corresponding signs flip each time two elements get swapped:

$$\mathrm{sign}(1,2,3) = 1, \quad \mathrm{sign}(1,3,2) = -1, \quad \mathrm{sign}(3,1,2) = 1,$$
$$\mathrm{sign}(3,2,1) = -1, \quad \mathrm{sign}(2,3,1) = 1, \quad \mathrm{sign}(2,1,3) = -1.$$

An even ~~A cyclic~~ **permutation** is a permutation σ with $\mathrm{sign}(\sigma) = 1$; an odd ~~a noncyclic~~ **permutation** is a permutation σ with $\mathrm{sign}(\sigma) = -1$.

EXAMPLES 2.18. Let $V = \mathbb{R}[x]_2$.

(1) Let $p, q \in \mathbb{R}[x]_2$. The function $\varphi(p,q) := p(\pi)q(33)$ is a bilinear form.
(2) Likewise,

$$\varphi(p,q) := p'(0)q(4) - 5p'(3)q''(\tfrac{1}{2})$$

is a bilinear form.

□

EXERCISE 2.19. Are the following functions bilinear forms?

(1) $V = \mathbb{R}^2$ and $\varphi(u,v) := \det \begin{bmatrix} u \\ v \end{bmatrix}$;
(2) $V = \mathbb{R}[x]_2$ and $\varphi(p,q) := \int_0^1 p(x)q(x)dx$;
(3) $V = M_{2\times 2}(\mathbb{R})$, the space of real 2×2 matrices, and $\varphi(L,M) := L_1^1 \operatorname{tr} M$, where L_1^1 it the (1,1)-entry of L and $\operatorname{tr} M$ is the trace of M;
(4) $V = \mathbb{R}^3$ and $\varphi(v,w) := v \times w$;
(5) $V = \mathbb{R}^2$ and $\varphi(v,w)$ is the area of the parallelogram spanned by v and w.

REMARK 2.20. We need to be careful about the following possible confusion. A bilinear form on V is a function on $V \times V$ that is linear in each variable *separately*. But $V \times V$ is also a vector space and one might wonder whether a bilinear form on V is also a linear form on the vector space $V \times V$. But this is not the case. For example consider the case in which $V = \mathbb{R}$, so that $V \times V = \mathbb{R}^2$ and let $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function:

(1) If $\varphi(x,y) := 2x - y$, then φ is not a *bilinear form on* $\mathbb{R}$, but is a *linear form* on $(x,y) \in \mathbb{R}^2$;
(2) If $\varphi(x,y) := 2xy$, then φ is a *bilinear form on* $\mathbb{R}$ (hence *linear* in $x \in \mathbb{R}$ and *linear* in $y \in \mathbb{R}$), but it is *not* a linear form on $\mathbb{R}^2$, as it is *not linear* in $(x,y) \in \mathbb{R}^2$.

So a *bi*linear form is not a form that it is "twice as linear" as a linear form, but a form that is defined on the product of twice the vector space.

□

EXERCISE 2.21. Verify the above assertions in Remark 2.20 to make sure you understand the difference.

NOTATION. We denote

$$\boxed{\mathrm{Bil}(V \times V, \mathbb{R}) = V^* \otimes V^*}$$

and call this vector space the **tensor product** of V^* and V^*.

REMARK 2.27. Just as it is for linear forms, to verify that two bilinear forms on V are the same it is enough to verify that they are the same on every pair of elements of ~~the~~ a basis. of V

$$\mathcal{B} \cancel{\times \mathcal{B} =} \{(b_i, \cancel{b_j}) : 1 \leq i, \cancel{j,} \leq n\}$$

of $V \times V$. In fact, let φ, ψ two bi linear forms and let us assume that

$$\varphi(b_i, b_j) = \psi(b_i, b_j)$$

for all $1 \leq i, j, \leq n$. Let $v = v^i b_i, w = w^j b_j \in V$ two vectors and let us verify that $\varphi(\cancel{v}\ v,w) = \psi(\cancel{v}\ v,w)$. In fact, because of the linearity in each variable,

$$\varphi(v, w) = \varphi(v^i b_i, w^j b_j) = v^i w^j \varphi(b_i, b_j) = v^i w^j \psi(b_i, b_j) = \psi(v^i b_i, w^j b_j) = \psi(v, w)\,.$$

□

PROOF OF PROPOSITION 2.26. The proof will be similar to the one of Proposition 2.9 for linear forms. We first check that the set of bilinear forms $\{\beta^i \otimes \beta^j,\ i, j = 1, \ldots, n\}$ consists of linearly independent vectors, then that it spans $\mathrm{Bil}(V \times V, \mathbb{R})$.

For the linear independence we need to check that the only linear combination of the $\beta^i \otimes \beta^j$ that gives the zero bilinear form is the trivial linear combination. Let $c_{ij}\beta^i \otimes \beta^j = 0$ be a linear combination of the $\beta^i \otimes \beta^j$. Then for all pairs of basis vectors (b_k, b_ℓ), with $k, \ell = 1, \ldots, n$, we have

$$0 = c_{ij}\beta^i \otimes \beta^j (b_k, b_\ell) = c_{ij}\delta^i_k \delta^j_\ell = c_{k\ell}\,,$$

thus showing the linear independence.

To check that $\mathrm{span}\{\beta^i \otimes \beta^j,\ i, j = 1, \ldots, n\} = \mathrm{Bil}(V \times V, \mathbb{R})$, we need to check that if $\varphi \in \mathrm{Bil}(V \times V, \mathbb{R})$, there exists $B_{ij} \in \mathbb{R}$ such that

$$\varphi = B_{ij}\beta^i \otimes \beta^j\,.$$

Because of (2.2) on p. 27, we obtain

$$\varphi(b_k, b_\ell) = B_{ij}\beta^i(b_k)\beta^j(b_\ell) = B_{ij}\delta^i_k\delta^j_\ell = B_{k\ell}\,,$$

for every pair $(b_k, b_\ell) \in V \times V$. Hence, we set $B_{k\ell} := \varphi(b_k, b_\ell)$. Now both φ and $\varphi(b_k, b_\ell)\beta^i \otimes \beta^j$ are bilinear forms and they coincide on $\mathcal{B} \times \mathcal{B}$. Because of the above remark, the two bilinear forms coincide. □

EXAMPLE 2.28. We continue with the study of the *scalar triple product* $\varphi_u : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, that was defined in Example 2.16 for a fixed given vector $u = \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$. We now want to find the components B_{ij} of φ_u with respect to the standard basis of $\mathbb{R}^3$.