

# Wahrscheinlichkeitstheorie und Statistik

## Lösungen Serie 2

Version 2 (24. Juni 2021: In Lösung 2.1(a) wurde die Notation für die Potenzmenge von  $2^\Omega$  auf  $\mathcal{P}(\Omega)$  geändert um mit der im Skript verwendeten Notation übereinzustimmen. Grundsätzlich werden beide Schreibweisen für die Potenzmenge in der Literatur verwendet.); Version 1 (28. Februar)

Bitte stellt Fragen in den Übungen und/oder stellt postet Fragen (anonym) in diesem file [https://docs.google.com/document/d/1AhLJZNTgsgkoC9Bmszy8kbYfTxFrEYznSXG1S1m\\_Ozc/edit?usp=sharing](https://docs.google.com/document/d/1AhLJZNTgsgkoC9Bmszy8kbYfTxFrEYznSXG1S1m_Ozc/edit?usp=sharing)

Wir empfehlen die Aufgaben selbständig zu lösen und dann auf <https://sam-up.math.ethz.ch/> hochzuladen oder selbst mit dieser Lösung zu vergleichen am besten rechtzeitig vor der Übung am **8. März**.

**Aufgabe 2.1** One coin is flipped and one die is rolled.

- (a) Define a suitable probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  space using a Laplace model.
- (b) Define random variables  $X : \Omega \rightarrow \mathbb{R}$  and  $Y : \Omega \rightarrow \mathbb{R}$  on this probability space such that  $X$  and  $Y$  represent the outcome of flipping the coin and of the roll of the die, respectively.
- (c) Show that  $\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x] \mathbb{P}[Y = y]$  for all  $x, y \in \mathbb{R}$ . (This means that the random variables  $X$  and  $Y$  are independent; see later.)

**Lösung 2.1**

- (a)  $\Omega := \{0, 1\} \times \{1, 2, 3, 4, 5, 6\}$ ,  $\mathcal{F} := \mathcal{P}(\Omega)$ ,  $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ ,  $A \mapsto \mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{|A|}{12}$ .
- (b)  $X : \Omega \rightarrow \mathbb{R}, \omega \mapsto X(\omega) := \omega_1$  and  $Y : \Omega \rightarrow \mathbb{R}, \omega \mapsto Y(\omega) := \omega_2$ .
- (c) First we compute the left-hand side as

$$\mathbb{P}[X = x, Y = y] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}] = \frac{|\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}|}{12},$$

where the numerator can be simplified as

$$|\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}| = \begin{cases} 1, & (x, y) \in \Omega, \\ 0, & (x, y) \notin \Omega. \end{cases}$$

This results in

$$\mathbb{P}[X = x, Y = y] = \begin{cases} \frac{1}{12}, & (x, y) \in \Omega, \\ 0, & (x, y) \notin \Omega. \end{cases}$$

Now, we consider the right-hand side by starting with calculating the probability

$$\mathbb{P}[X = x] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x\}] = \frac{|\{\omega \in \Omega : X(\omega) = x\}|}{12},$$

where the set appearing in the numerator can be expressed as

$$\{\omega \in \Omega : X(\omega) = x\} = \begin{cases} \{(x, 1), (x, 2), (x, 3), (x, 4), (x, 5), (x, 6)\}, & x \in \{0, 1\}, \\ \emptyset, & x \notin \{0, 1\}. \end{cases}$$

So by counting, we obtain

$$\mathbb{P}[X = x] = \begin{cases} \frac{6}{12} = \frac{1}{2}, & x \in \{0, 1\}, \\ 0, & x \notin \{0, 1\}. \end{cases}$$

Analogously, we get

$$\mathbb{P}[Y = y] = \begin{cases} \frac{2}{12} = \frac{1}{6}, & y \in \{1, 2, 3, 4, 5, 6\}, \\ 0, & y \notin \{1, 2, 3, 4, 5, 6\}. \end{cases}$$

Since  $(x, y) \in \Omega$  is equivalent to  $(x \in \{0, 1\}) \wedge (y \in \{1, 2, 3, 4, 5, 6\})$ , we can conclude that

$$\mathbb{P}[X = x] \mathbb{P}[Y = y] = \begin{cases} \frac{1}{2} \frac{1}{6} = \frac{1}{12}, & (x, y) \in \Omega, \\ 0, & (x, y) \notin \Omega. \end{cases}$$

**Aufgabe 2.2** We have two dice. One is ordinary with the numbers 1, 2, 3, 4, 5, 6 and one is special where 6 is replaced by 7 (i.e. 1, 2, 3, 4, 5, 7). We flip a coin to decide which die is rolled. If flipping the coin results in heads, the ordinary die is rolled, otherwise the special die is rolled.

- (a) Define a suitable probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  using a Laplace model.
- (i) Define random variables  $X : \Omega \rightarrow \mathbb{R}$  and  $Y : \Omega \rightarrow \mathbb{R}$  such that  $X$  and  $Y$  represent the outcomes of flipping the coin and of the roll of the die, respectively.
- (ii) What is the cardinality  $|\mathcal{F}|$ ? Give examples of events  $E_1, E_2, E_3, E_4 \in \mathcal{F}$  such that  $\mathbb{P}[E_i] \neq \mathbb{P}[E_j], \forall i \neq j$ .
- (b) What is the probability that rolling the die results in an even number?
- (c) Show that  $\mathbb{P}[X = x, Y = y] \neq \mathbb{P}[X = x]\mathbb{P}[Y = y]$  for some  $x, y \in \mathbb{R}$ . (This means that the random variables  $X$  and  $Y$  are not independent; see later and cf. [Aufgabe 2.1](#).)

### Lösung 2.2

- (a)  $\Omega := \{0, 1\} \times \{1, 2, 3, 4, 5, 6\}$ ,  $\mathcal{F} := 2^\Omega$ ,  $\mathbb{P} : \mathcal{F} \rightarrow [0, 1], A \mapsto \mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{|A|}{12}$ . Alternatively one could define  $\Omega := \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 7)\}$ . Then  $Y$  can be defined more simply as  $Y(\omega) := \omega_2$ .

$$(i) \quad X : \Omega \rightarrow \mathbb{R}, \omega \mapsto X(\omega) := \omega_1 \text{ and } Y : \Omega \rightarrow \mathbb{R}, \omega \mapsto Y(\omega) := \begin{cases} \omega_2, & \omega_2 \leq 5, \\ 6, & \omega_2 = 6 \text{ and } \omega_1 = 0, \text{ An} \\ 7, & \omega_2 = 6 \text{ and } \omega_1 = 1. \end{cases}$$

elegant way to write this is  $Y(\omega) := \omega_2 I_{\{\omega_2 \leq 5\}} + (\omega_1 + \omega_2) I_{\{\omega_2 > 5\}}$ .

- (ii)  $|\mathcal{F}| = 2^{12} = 4096$ , so there are many different possibilities to choose the examples—e.g.  $E_1 = \emptyset, E_2 = \{(0, 1)\}, E_3 = \{(0, 1), (0, 3), (1, 2)\}, E_4 = \Omega$  with  $\mathbb{P}[E_1] = 0, \mathbb{P}[E_2] = \frac{1}{12}, \mathbb{P}[E_3] = \frac{3}{12}, \mathbb{P}[E_4] = 1$ .
- (b) We can directly count that  $|\{\omega \in \Omega : \exists k \in \mathbb{N} \text{ with } Y = 2k\}| = 3 + 2 = 5$ . Hence, the probability is  $\mathbb{P}[\{\omega \in \Omega : \exists k \in \mathbb{N} \text{ with } Y = 2k\}] = \frac{5}{12}$ .
- (c) To show the dependence of the two random variables, it is sufficient to find one counterexample for the equation. A simple choice is  $x = 0$  and  $y = 7$ . First we compute the left-hand side as

$$\mathbb{P}[X = 0, Y = 7] = \frac{|\{\omega \in \Omega : X(\omega) = 0 \text{ and } Y(\omega) = 7\}|}{12} = 0.$$

Now, we consider the right-hand side by starting with calculating the probability

$$\mathbb{P}[X = 0] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = 0\}] = \frac{|\{\omega \in \Omega : X(\omega) = 0\}|}{12} = \frac{6}{12} = \frac{1}{2}.$$

Analogously, we get

$$\mathbb{P}[Y = 7] = \mathbb{P}[\{\omega \in \Omega : Y(\omega) = 7\}] = \frac{|\{\omega \in \Omega : Y(\omega) = 7\}|}{12} = \frac{1}{12}.$$

So we can conclude that

$$\mathbb{P}[X = 0, Y = 7] = 0 \neq \frac{1}{24} = \mathbb{P}[X = 0]\mathbb{P}[Y = 7].$$

### Aufgabe 2.3

- (a) Seien  $A$  und  $B$  zwei Ereignisse mit

$$\mathbb{P}[A^c] = \frac{1}{2}, \quad \mathbb{P}[B^c] = \frac{1}{2}, \quad \mathbb{P}[A^c \cap B^c] = p.$$

Bestimmen Sie als Funktion von  $p$

- (i) die Wahrscheinlichkeiten  $\mathbb{P}[A \cap B], \mathbb{P}[A \cap B^c]$  und  $\mathbb{P}[A^c \cap B]$ . In welchem Bereich darf  $p$  liegen?
- (ii) die Wahrscheinlichkeit, dass höchstens  $i$  der beiden Ereignisse  $A$  und  $B$  eintreten, wobei  $i = 0, 1, 2$ .

**Lösung 2.3**

(a) (i) Da

$$(A \cup B)^c = (A^c \cap B^c),$$

bekommt man mit Hilfe der Formel des Skripts

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \\ &= [1 - \mathbb{P}(A^c)] + [1 - \mathbb{P}(B^c)] - [1 - \mathbb{P}(A^c \cap B^c)] \\ &= \frac{1}{2} + \frac{1}{2} - (1 - p) \\ &= p. \end{aligned}$$

Da  $A \cap B^c$  und  $A \cap B$  disjunkt sind mit  $(A \cap B^c) \cup (A \cap B) = A$ , erhalten wir

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = \frac{1}{2} - p.$$

Ähnlich bekommt man

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{1}{2} - p.$$

Da alle Wahrscheinlichkeiten in  $[0, 1]$  liegen müssen, muss  $p$  im Bereich  $[0, \frac{1}{2}]$  liegen.

(ii) Wir definieren das Ereignis

$$C_i := \text{„Höchstens } i \text{ der beiden Ereignisse } A \text{ und } B \text{ treten ein.“},$$

für alle  $i = 0, 1, 2$ . Wir erhalten

$$\mathbb{P}(C_0) = \mathbb{P}(A^c \cap B^c) = p,$$

$$\begin{aligned} \mathbb{P}(C_1) &= \mathbb{P}(C_0) + \mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(C_0) + 1 - \mathbb{P}(A^c \cap B^c) - \mathbb{P}(A \cap B) \\ &= p + 1 - p - p \\ &= 1 - p \end{aligned}$$

und

$$\mathbb{P}(C_2) = \mathbb{P}(C_1) + \mathbb{P}(A \cap B) = 1 - p + p = 1.$$

Wenn du Feedback zum Übungszettel hast, schreibe bitte eine Mail an [Jakob Heiss](#).